

Second Order Stochastic Dominance

Definition

Suppose the random variables X and Y have support on $[l, u]$. Then X *second-order stochastically dominates* Y if

$$\int_l^a \Pr[X > t]dt \geq \int_l^a \Pr[Y > t]dt$$

for all a .

Results

1. If X first order stochastically dominates Y , then X second-order stochastically dominates Y .

Proof. If X first order stochastically dominates Y , then

$$\Pr[X > t] \geq \Pr[Y > t]$$

for all t . Integrating both sides over t gets the desired result.

2. If X second-order stochastically dominates Y , then $E[X] \geq E[Y]$.

Proof. Recall that

$$E[X] = l + \int_l^u [1 - F(t)]dt$$

By second order stochastic dominance, this is greater than

$$E[Y] = l + \int_l^u [1 - G(t)]dt$$

3. X second-order stochastically dominates Y if and only if

$$E[h(X)] \geq E[h(Y)]$$

for all increasing and concave function h .

Proof. Let

$$h(t) = \begin{cases} t & \text{if } t \leq a \\ a & \text{if } t > a \end{cases}$$

Obviously h is increasing and concave. Now,

$$\begin{aligned} E[h(X)] &= \int_l^a tf(t)dt + \int_a^u af(t)dt \\ &= [-t(1 - F(t))]_{t=l}^{t=a} + \int_l^a [1 - F(t)]dt + a(1 - F(a)) \\ &= \int_l^a [1 - F(t)]dt + l \end{aligned}$$

Similarly,

$$E[h(Y)] = \int_l^a [1 - G(t)]dt + l$$

Hence if $E[h(X)] \geq E[h(Y)]$, then X second-order stochastically dominates Y .

Next, we need to prove the reverse statement. Let $X' = h(X)$ and $Y' = h(Y)$, where h is any increasing and concave function. Let $l' = h(l)$ and $a' = h(a)$. We have

$$\begin{aligned} \int_{l'}^{a'} \Pr[X' > x]dx &= \int_{l'}^{a'} \Pr[X > h^{-1}(x)]dx \\ &= \int_{l'}^{a'} [1 - F(h^{-1}(x))]dx \end{aligned}$$

Let $t = h^{-1}(x)$. Then $h(t) = x$. Hence, $h'(t)dt = dx$. Using the method of substitution, the integral is equal to

$$\int_l^a (1 - F(t))h'(t)dt$$

Let $F^*(a) = \int_l^a (1 - F(t))dt$. Using integration by parts, this is equal to

$$[h'(t)F^*(t)]_{t=l}^{t=a} - \int_l^a h''(t)F^*(t)dt$$

Since $F^*(l) = 0$, this is equal to

$$h'(a)F^*(a) - \int_l^a h''(t)F^*(t)dt$$

By second-order stochastic dominance, $F^*(t) \geq G^*(t)$ for all t . Furthermore, since $h'(a) \geq 0$ and $h''(t) \leq 0$, the above is greater than

$$\int_{l'}^{a'} \Pr[Y' > y]dy = h'(a)G^*(a) - \int_l^a h''(t)G^*(t)dt$$

for all a' . Hence, X' second-order stochastically dominates Y' . By result 2, we conclude $E[h(X)] \geq E[h(Y)]$.

4. If X second-order stochastically dominates Y , and if X and Y have the same mean, then $E[h(X)] \geq E[h(Y)]$ for all concave function h (notice that h does not have to be increasing).

Proof. From the proof of result 3, we have

$$E[h(X)] = h(l) + h'(u)F^*(u) - \int_l^u h''(t)F^*(t)dt$$

Similarly,

$$E[h(Y)] = h(l) + h'(u)G^*(u) - \int_l^u h''(t)G^*(t)dt$$

But since $F^*(u) = G^*(u)$ if X and Y have the same mean, and since $F^*(t) \geq G^*(t)$, we have

$$E[h(X)] - E[h(Y)] = \int_l^u h''(t)(G^*(t) - F^*(t))dt \geq 0$$

5. If X second-order stochastically dominates Y , and if X and Y have the same mean, then X has a smaller variance than Y .

Proof.

$$\text{Var}[X] = E[X^2] - E[X]^2$$

Since X^2 is convex, result 4 implies that $E[X^2] \leq E[Y^2]$.