Strategic Delay in a Real Options Model of R&D Competition

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This paper considers irreversible investment in competing research projects with uncertain returns under a winner-takes-all patent system. Uncertainty takes two distinct forms: the technological success of the project is probabilistic, while the economic value of the patent to be won evolves stochastically over time. According to the theory of real options uncertainty generates an option value of delay, but with two competing firms the fear of preemption would appear to undermine this approach. In non-cooperative equilibrium two patterns of investment emerge depending on parameter values. In a pre-emptive leader–follower equilibrium firms invest sequentially and option values are reduced by competition. A symmetric outcome may also occur, however, in which investment is more delayed than the single-firm counterpart. Comparing this with the optimal cooperative investment pattern, investment is found to be more delayed when firms act non-cooperatively as each holds back from investing in the fear of starting a patent race. Implications of the analysis for empirical and policy issues in R&D are considered.

1. INTRODUCTION

When a firm has the opportunity to make an irreversible investment facing future uncertainty there is an option value of delay. By analogy with a financial call option it is optimal to delay exercising the option to invest, even when it would be profitable to do so at once, in the hope of gaining a higher payoff in the future. Using this insight the real options approach improves upon traditional NPV-based investment appraisal methods by allowing the value of delay and the importance of flexibility to be quantified and incorporated explicitly into the analysis.

Real options, unlike their financial counterparts, are rarely backed by legal contracts guaranteeing the holder’s rights in precise terms. Most are non-proprietary investment opportunities whose terms are somewhat vague and far from guaranteed. In particular, a firm’s ability to hold the option is frequently influenced by the possibility that another firm may exercise a related option which affects its value. In a few instances a legally enforceable property right such as an oil lease or patent confers a proprietary right similar to that granted by a financial option. Or occasionally a firm has such a strong market position, such as in a natural monopoly or network industry, that its investment opportunities are de facto proprietary. But in most industries some degree of competition, either actual or potential, exists and the option to invest cannot be held independently of strategic considerations.

When related options are held by a small number of firms with an advantage to the first mover, each firm’s ability to delay is undermined by the fear of preemption. Consider the case where two firms have the ability to exercise an option and the first to do so receives the underlying asset at once, leaving the second mover empty-handed. Each would like to exercise the option just before its rival, giving rise to discontinuous Bertrand-style reaction functions. The option will be exercised as soon as the payoff from doing so becomes marginally positive and the value of delay is eliminated. Under such circumstances the real options approach is irrelevant and the traditional NPV rule resurfaces as the appropriate method of investment appraisal.
In order to study the tension between real options and strategic competition this paper adapts the continuous time framework of Fudenberg and Tirole (1985) in two important respects to apply to the specific context of competing investments in R&D. The firms' profit functions are specified so as to include two distinct forms of uncertainty: economic uncertainty over the future profitability of the project and technological uncertainty over the success of R&D investment itself. Economic uncertainty gives rise to option values and a tendency for delay. Meanwhile the winner-takes-all nature of the patent system generates a first-mover advantage that counteracts the incentive to delay. However, technological uncertainty over the outcome of R&D drives a wedge between a firm's decision to invest and the outcome of that investment, mitigating the preemption incentive since the first mover is no longer guaranteed to win the patent.\(^1\) Option values can therefore be preserved, at least to some extent. The instantaneous probability of success, or hazard rate, captures in a simple form the strength of the first-mover advantage, allowing outcomes for varying degrees of preemption to be readily compared. A high hazard rate means that the breakthrough is likely to occur soon after research commences, worsening the pre-emption effect.

Focusing on Markov perfect equilibria the outcome of the non-cooperative two-player game is found to take one of two forms depending upon parameter values. One is a pre-emptive leader–follower outcome in which one firm invests strictly before its rival and option values are substantially undermined. The other has a multiplicity of equilibria including a continuum of symmetric equilibria in which the firms invest at the same trigger point. The Pareto-dominant equilibrium coincides with the optimal joint-investment rule which would be chosen by firms that agree to adopt a common trigger point. This outcome entails greater delay than the single-firm counterpart.

The optimal cooperative investment rule for two firms (or research units of the same firm) is derived as a benchmark. This is found to involve sequential investment so that research efforts are phased in over time. Compared with the pre-emptive leader–follower equilibrium, the cooperative trigger points are higher than their non-cooperative counterparts. The joint-investment equilibrium, although preferable to the leader–follower outcome, is also shown to be sub-optimal. It is, however, the second-best optimum when firms are constrained to choose a symmetric investment rule. Comparing this with the cooperative pattern, strategic interactions are seen to increase the time to first investment; this is in stark contrast to the usual presumption that the fear of preemption speeds up investment.

The joint-investment equilibrium can be understood by considering the effects of R&D competition. By reducing the value of investment to the firm that moves later, the possibility of rival innovation creates a first-mover advantage that tends to induce earlier investment. But the threat of innovation by the leader also reduces the value of its rival's option to delay, speeding up the competitive reaction to the leader's investment. Anticipating this reaction a firm may prefer to delay its investment. In effect, an investing firm chooses the time at which the patent race will begin and it is better for each if this is delayed until the optimal joint-investment point. A useful analogy is the behaviour of contestants in a long-distance race, who typically remain in a pack proceeding at a moderate pace for most of the distance, until near the end when one attempts to break away and the sprint for the finish begins. This interaction between option values and preemption contrasts with existing models combining real options and strategic interactions, such as Smets (1991), in which the roles of option values and competition are additive. In these models rivalry merely reduces the value of investment, with the option value of delay remaining unchanged.

\(^1\) It should be noted that the advantage gained by the first mover is not a persistent one: if the breakthrough is not achieved before the follower invests the two firms are equally likely to succeed from then on.
By combining irreversible investment under uncertainty with strategic interactions in the presence of technological uncertainty, this paper brings together three branches of the literature. Real options models have been used to explain delay and hysteresis arising in a number of contexts, but these are mostly set in a monopolistic or perfectly competitive framework: see, for example, McDonald and Siegel (1986), Pindyck (1988) and Dixit (1989, 1991). The second branch analyses timing games of entry and exit in a deterministic framework: these are straightforward stopping time games where the underlying process is simply time itself. Fudenberg et al. (1983) and Fudenberg and Tirole (1985) model pre-emption games, while wars of attrition are modelled by Ghemawat and Nalebuff (1985) and Fudenberg and Tirole (1986). Last, technological uncertainty is modelled as a Poisson arrival in papers by, inter alia, Loury (1979), Dasgupta and Stiglitz (1980), Lee and Wilde (1980), Reinganum (1983) and Dixit (1988). These papers, however, assume the return to successful R&D (or demand in the product market from which it is derived) to be deterministic, thus ruling out any option value of delay and the related timing issues.

Literature combining real options with strategic interactions is fairly limited. Smets (1991) examines irreversible market entry for a duopoly facing stochastic demand. Non-cooperative behaviour results in an asymmetric leader–follower equilibrium. When the leadership role is exogenously pre-assigned the cooperative symmetric outcome may then be attained. Grenadier (1996) considers the strategic exercise of options in real estate markets; joint investment arises only when the stochastic process starts at a sufficiently high value and even then does not necessarily occur at the optimal point. In a two-player game where each player’s exercise cost is private information, Lambrecht and Perraudin (2002) find trigger points located intermediate between the monopoly and simple NPV outcomes. Kulatilaka and Perotti (1998) consider the effect of uncertainty on the value of strategic investment.

The paper is structured as follows. The model is described in Section 2. The optimisation problem of a single firm facing no actual or potential competition is solved in Section 3. Section 4 derives the optimal cooperative investment plan for two firms. Non-cooperative equilibrium in the two-player game is found in Section 5. The findings are discussed in Section 6; Section 7 concludes.

## 2. THE MODEL

Two risk-neutral firms, \( i = 1, 2 \), have the opportunity to invest in competing research projects. Research is directly competitive: the firms strive for the same patent and successful innovation by one eliminates all possible profit for the other. The firms face both technological and economic uncertainty. Discovery by an active firm is a Poisson arrival, while the value of the patent received by the successful inventor evolves stochastically over time.\(^2\) The decision to invest in a research project is assumed to be irreversible. The possible states of firm \( i \) are denoted \( \theta_i \in \{0, 1\} \) for the idle and active states respectively.

The value of the patent, \( \pi \), evolves exogenously and stochastically according to a geometric Brownian motion (GBM) with drift given by the following expression:

\[
d\pi_t = \mu \pi_t dt + \sigma \pi_t dW
\]

where \( \mu \in [0, r) \) is the drift parameter measuring the expected growth rate of \( \pi \),\(^3\) \( r \) is the risk-free interest rate, assumed to be constant over time, \( \sigma > 0 \) is the instantaneous standard

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2. This value could be interpreted as the expected NPV of profits in the relevant product market or, if further sunk costs are required, might itself be the value of the option to invest in this market, making investment in the research stage a compound option problem.

3. The restriction that \( \mu < r \), commonly found in real options models, is necessary to ensure that there is a strictly positive opportunity cost to holding the option, so that it will not be held indefinitely. A large negative drift term
deviation or volatility parameter, and \( dW \) is the increment of a standard Wiener process where \( dW \sim N(0, dt) \).

Each firm has the opportunity to invest in a research project. Following Loury (1979), firm \( i \) sets up a research project by investing an amount \( K_i > 0 \). From the time of this investment, discovery takes place randomly according to a Poisson distribution with constant hazard rate \( h_i > 0 \). Thus the hazard rate is independent of the duration of research and the number of firms investing; possible variations on this assumption are discussed in Section 7. The probabilities of discovery by each firm are independent. We focus on the symmetric case where \( h_i = h \) and \( K_i = K \) for \( i = 1, 2 \). All parameter values and actions are common knowledge, thus the game is one of complete information.

The following assumptions are made:

**Assumption 1.** \( E_0(\int_0^\infty e^{-(r+h)t} h \pi_i dt) - K < 0 \).

**Assumption 2.** If \( \theta_i(\tau) = 1 \) then \( \theta_i(t) = 1 \) \( \forall t \geq \tau \).

Assumption 1 states that the initial value of the patent, \( \pi_0 \), is sufficiently low that the expected return from immediate investment is negative, ensuring that neither firm will invest at once. Assumption 2 formalizes the irreversibility of investment and constrains the strategy of the firm accordingly: if firm \( i \) has already invested by date \( \tau \) then it remains active at all dates subsequent to \( \tau \) until the game ends with a discovery.

In a multi-agent setting the firm’s investment problem can no longer be solved using the optimisation techniques typically employed in real options analysis. Instead, the optimal control problem becomes a stopping time game; for a detailed analysis see Dutta and Rustichini (1993). In a stopping time game each player has an irreversible action such that following an action by one or more players expected payoffs in the subsequent subgame are fixed. Dutta and Rustichini allow for the possibility that, as in this paper, the stochastic process continues to evolve after the leader’s action while the follower still has a move to make. The stopping time game is described by the stochastic process \( \pi \) and the payoff functions for the leader and follower, derived in Section 5 below.

The game proceeds as follows. In the absence of an action taken by either firm, the stochastic process evolves according to (1). If firm \( i \) has not commenced research at any time \( \tau < t \) its action set is \( A_i^\tau = \{ \text{invest, don’t invest} \} \). If, on the other hand, \( i \) has invested at some \( \tau < t \), then \( A_i^\tau \) is the null \{don’t move\}. Thus each firm faces a control problem in which its only choice is when to choose the action ‘stop’—or rather, in this case, to commence research. After taking this action the firm can make no more moves and cannot further influence the outcome of the game. The game ends when a discovery is made by either firm.

A strategy for firm \( i \) is a mapping from the history of the game \( H_t \) to the action set \( A_i^t \) as follows: \( \sigma_i^t : H_t \rightarrow A_i^t \). At time \( t \geq 0 \), the history of the game has two components, the sample path of the stochastic state variable \( \pi \) and the actions of the two firms up to date \( t \). With irreversible investment the history of actions in the game at \( t \) is summarized by the fact that the game is still continuing at \( t \) (i.e. \( \theta_i = 0 \) \( \forall i \)). However, the history of the state variable is more complex since its current value could have been reached by any one of a huge number of possible paths.

would, ceteris paribus, encourage earlier investment to raise the probability of winning the prize before its value declines significantly, counteracting the option effects in the model. To avoid such an outcome we make the assumption that \( \mu \) is non-negative. Since the model is concerned with the effects of uncertainty, not expected trends, the conclusions from the analysis are unaffected by this assumption.

4. The R&D cost is fixed, or contractual in the terminology of Kamien and Schwartz (1982).
Firms are assumed to employ Markov strategies: actions are functions of the current state alone and the strategy formulation does not vary with time. Since \( \pi \) follows a Markov process, Markov strategies incorporate all payoff-relevant factors in the game. Furthermore, if one player uses a Markov strategy then its rival has a best response that is Markov as well; hence a Markov equilibrium remains an equilibrium even when history-dependent strategies are permitted, although other non-Markov equilibria may then also exist. (For further explanation see Maskin and Tirole (1988) and Fudenberg and Tirole (1991, Chapter 13).) A player’s strategy is a stopping rule specifying a critical value or “trigger point” for the stochastic variable \( \pi \) at which the firm invests.\(^5\)

As Fudenberg and Tirole (1985) point out, the use of continuous time complicates the formulation of strategies as there is a loss of information inherent in taking the limit of a discrete time mixed strategy equilibrium. To deal with this problem they extend the strategy space to include not only the cumulative probability that a player has adopted, but also the “intensity” with which a player adopts “just after” the cumulative probability has jumped to one. Although this formulation uses symmetric mixed strategies, equilibrium outcomes are equivalent to those in which firms employ pure strategies and may adopt asymmetric roles.\(^6\) Thus, although the underlying framework is an extended space with symmetric mixed strategies, the analysis will proceed as if each firm uses a (possibly asymmetric) pure Markov strategy.

### 3. OPTIMAL INVESTMENT TIMING FOR A SINGLE FIRM

The optimal investment rule for a single firm facing no competition is found by solving the stochastic optimal stopping problem

\[
V(\pi_t) = \max_T E_t \left\{ e^{-rT} \left( \int_T^\infty e^{-(r+h)\tau} h\pi_t d\tau - K \right) \right\}
\]

where \( E_t \) denotes expectations conditional on information available at time \( t \) and \( T \) is the future stopping time at which the investment is made. This problem is solved as follows (for more details see Dixit and Pindyck (1994), Harrison (1985)). First, expressions for the value of the firm in the continuation and stopping regions are derived by solving the relevant Bellman equations. In the continuation region the firm holds the option to invest, analogous to a call option, while in the stopping region the irreversibility of investment entails that its value is simply the expected value of the project. Next the boundary between these regions, the optimal stopping point \( \pi_U \), is found by imposing value-matching and smooth-pasting conditions.\(^7\)

The single firm’s value function is given by

\[
V_U(\pi) = \begin{cases} 
B_0 \pi^{\beta_0} h\pi \quad & \text{for } \pi < \pi_U \\
\frac{h\pi}{r+h-\mu} - K & \text{for } \pi \geq \pi_U
\end{cases}
\]

5. To be precise, the statement that a firm invests at a trigger point \( \pi^* \) means that the firm invests at the time when the stochastic process \( \pi \) first crosses the value \( \pi^* \), approaching this level from below.

6. For further details see Fudenberg and Tirole (1985, Section 4).

7. The smooth-pasting condition requires the value functions to meet at the trigger point with equal first derivatives. To see that this is necessary for optimality, suppose instead that a kink arose at \( \pi_U \). By delaying for a short interval after \( \pi_U \) was first reached the next step \( d\pi \) could be observed. If the kink was convex, the firm would obtain a higher expected payoff by entering if and only if \( \pi \) moves strictly above \( \pi_U \), since an average of points on either side of the kink gives a higher expected value. If the kink were concave second order conditions would be violated. A more detailed explanation of this condition can be found in Appendix C of chapter 4 in Dixit and Pindyck (1994). Note that this condition applies for all diffusion processes, not just a GBM such as (1).
where $B_0 = \frac{h\pi^{1-\beta_0}}{(r+h-\mu)\beta_0}$ and $\beta_0 = \frac{1}{2}\left[1 - \frac{2\mu}{\sigma^2} + \sqrt{(1 - \frac{2\mu}{\sigma^2})^2 + \frac{8r}{\sigma^2}}\right]$; $1 < \beta_0 < \frac{r}{\mu}$.

Note that the hazard rate enters the denominator of the expected value term in the form of an augmented discount rate, $r + h$. This result is typical of models involving a Poisson arrival process: for similar examples in the context of R&D see Loury (1979), Dasgupta and Stiglitz (1980), Lee and Wilde (1980) and Dixit (1988).

In the absence of time dependence the optimal investment rule is a trigger value $\pi_U$ such that the firm delays while $\pi < \pi_U$ and invests immediately for $\pi \geq \pi_U$. This trigger point is given by

$$\pi_U = \frac{\beta_0}{(\beta_0 - 1)} \frac{(r + h - \mu)}{h} K.$$  \hspace{1cm} (4)

4. THE COOPERATIVE BENCHMARK

We next consider the benchmark case in which the two firms (or research units) plan their investments cooperatively.\(^9\) We start by supposing, without loss of generality, that one unit invests at a trigger point $\pi_1$ and the other when a second trigger $\pi_2 \geq \pi_1$ is reached. The value of the combined entity under this investment plan is described by

$$V_{L+F}(\pi) = \begin{cases} A_0 \pi^2 \beta_0 h \pi r + h - \mu + A_1 \pi^2 \beta_1 - K & \text{for } \pi < \pi_1 \\ 2\text{NPV}(\pi) - 2K & \text{for } \pi \geq \pi_2 \end{cases}$$  \hspace{1cm} (5)

where $\beta_1 = \frac{1}{2}\left[1 - \frac{2\mu}{\sigma^2} + \sqrt{(1 - \frac{2\mu}{\sigma^2})^2 + \frac{8r}{\sigma^2}}\right] > \beta_0$ and $\text{NPV}(\pi) = \frac{h\pi}{r+2h-\mu}$, the expected NPV of a research unit when the other unit is also active.

The optimal choice of $\pi_1$ and $\pi_2$, along with the option value terms $A_0$ and $A_1$, is determined by imposing value-matching and smooth-pasting conditions between the relevant components of the value function at each trigger point.\(^10\) Solving these conditions at $\pi_2$ yields

$$\pi_2 = \frac{\beta_1}{(\beta_1 - 1)} \frac{(r + 2h - \mu)(r + h - \mu)}{h(r - \mu)} K$$  \hspace{1cm} (6)

and

$$A_1 = \frac{K\pi_2^{-\beta_1}}{(\beta_1 - 1)} > 0.$$  \hspace{1cm} (7)

Imposing optimality conditions at $\pi_1$ and substituting for $A_1$ yields the following implicit expression for $\pi_1$:

$$\left(\beta_0 - 1\right) \frac{h\pi_1}{(r + h - \mu)} - \left(\frac{\beta_1 - \beta_0}{\beta_1 - 1}\right) K \left(\frac{\pi_1}{\pi_2}\right)^{\beta_1} - \beta_0 K = 0.$$  \hspace{1cm} (8)

As it stands there is nothing to prevent the two units investing simultaneously, \(i.e.\) the formulation does not preclude the solution $\pi_1 = \pi_2$. We now wish to determine whether the cooperative optimum entails simultaneous or sequential investment. Lemma 1 forms the basis for this analysis.

\(^8\) See Dixit and Pindyck (1994, p. 189) for a proof of this upper bound.

\(^9\) It is implicitly assumed that side payments may be used to ensure that neither firm has an incentive to deviate; alternatively, the two firms may be separate research units controlled by an integrated firm.

\(^10\) In contrast, as will be seen in the next section, no smooth-pasting condition obtains at the leader’s trigger point in the non-cooperative case.
Lemma 1. Over the interval \([0, \pi_2]\) equation (8) has a unique root, \(\pi_1 \in (0, \pi_2)\).

Proof. See appendix. ||

Proposition 1 follows directly from Lemma 1.

Proposition 1. The cooperative optimum is uniquely defined as a sequential investment pattern in which one research unit invests at \(\pi_1\) and the other invests strictly later at \(\pi_2\), where these trigger points satisfy (8) and (6) respectively.

Thus, two cooperating firms would choose to phase in their R&D investments progressively over time rather than invest both units at once. The sequential pattern gives some possibility of a return even when the patent value is low (though NPV-positive), reducing the opportunity cost of delay, while holding back from committing all investment costs at once and retaining the option to expand R&D in the future.

It is convenient at this point to derive the second-best optimum when the two firms are constrained to invest at the same trigger point, due perhaps to difficulties in agreeing an asymmetric pattern or making side payments. The optimal symmetric investment rule follows straightforwardly from the analysis of Section 3, being equivalent to the single firm problem with an investment cost of \(2K\) and arrival rate \(2h\). Denoting the optimal joint-investment trigger point as \(\pi_C\), this analysis yields

\[
\pi_C = \frac{\beta_0}{(\beta_0 - 1)} \frac{(r + 2h - \mu)}{h} K.
\]  

(9)

The value of an individual firm under this scenario is described by the following value function (i.e. an integrated firm consisting of two research units has twice this value):

\[
V_C(\pi) = \begin{cases} 
  B_C \pi^{\beta_0} & \text{for } \pi < \pi_C \\
  \text{NPV}(\pi) - K & \text{for } \pi \geq \pi_C
\end{cases}
\]

(10)

where \(B_C = \frac{h\pi_C^{1-\beta_0}}{(r+2h-\mu)\beta_0}\).

Comparing (9) with (4) it can readily be seen that \(\pi_C > \pi_U\), thus investment takes place strictly later when two firms agree a common investment rule than when a single firm acts alone. Note that this result is due to the indirect effect of the hazard rate on the augmented discount rate: since both the cost and hazard rate of research are doubled there is no direct effect on the efficiency of R&D. Comparing \(\pi_C\) with the unconstrained optimum it can be demonstrated that the constrained optimum lies between the two unconstrained trigger points.

Lemma 2. \(\pi_2 > \pi_C\) for \(h > 0\).

Lemma 3. \(\pi_C > \pi_1\) for \(h > 0\).

Proofs. See appendix. ||

Proposition 2. The ranking of trigger points in the optimal cooperative investment plan and the constrained-optimal joint-investment rule is given by \(\pi_1 < \pi_C < \pi_2\).
5. NON-COOPERATIVE EQUILIBRIUM

We turn now to the non-cooperative two-player game. We start by assuming, without loss of generality, that one firm (the leader) invests strictly before its rival (the follower). As usual in dynamic contexts the stopping time game is solved backwards; we therefore start by considering the optimization problem of the follower.

5.1. The follower’s investment problem

Given that the leader has already sunk its investment, the follower faces the conditional probability that its rival will innovate first. With independent hazard rates this probability is the same regardless of whether the follower itself has or has not invested; thus the follower’s investment problem is equivalent to that of a single firm with the augmented discount rate \( r + h \). This decision problem can be solved following the analysis of Section 3 but replacing \( r \) by \( r + h \) throughout, to yield the follower’s trigger point

\[
\pi_F = \frac{\beta_1}{(\beta_1 - 1)} \frac{(r + 2h - \mu)}{h} K
\]  

(11)

where \( \beta_1 \) is as defined following expression (5). The follower’s value function is described by

\[
V_F(\pi) = \begin{cases} 
B_F \pi^{\beta_1} & \text{for } \pi < \pi_F \\
NPV(\pi) - K & \text{for } \pi \geq \pi_F 
\end{cases}
\]  

(12)

where \( B_F = \frac{h\pi_F^{1-\beta_1}}{(r+2h-\mu)\beta_1} > 0 \).

Comparing \( \pi_F \) with the triggers derived in the previous two sections it can readily be seen that \( \pi_F < \pi_C \). However, \( \pi_F \) and \( \pi_U \) cannot be ranked in general since the leader’s hazard rate has two conflicting effects on the follower. The direct effect of the leader’s research activity is to reduce the expected value of investment to the follower. On its own the lower value of investment raises \( \pi_F \), causing the follower to act later compared with the single-firm case. However, there is a second effect via the option value mark-up factor involving \( \beta_1 \), which is smaller than the corresponding factor for the single firm. As explained in the introduction, the threat of rival innovation reduces the follower’s option value of delay, tending to hasten its investment. The first effect is found in the duopoly models of Smets (1991) and Grenadier (1996) where competition similarly reduces the value of investment to the follower. However the second, option value effect of the leader’s investment is absent from these models.

5.2. The leader’s payoff

We now derive the payoff to a firm that invests as the leader given that the follower acts optimally in the future. Once the leader invested it has no further decision to make and its payoff is given by the expected value of its research project. However, this payoff is likely to be reduced in the future by the follower’s investment at \( \pi_F \). The leader’s value function therefore has two distinct segments: its value before the follower invests and its value after this investment takes place.

After the follower invests, the leader’s (and the follower’s) value is given by the expected value \( NPV(\pi) \), defined following (5). Before the follower invests the leader’s value function has two components: the expected value of research and an option-like term that anticipates the negative effect of the follower’s investment on the leader’s expected payoff. The constant term in this option-like component is found by imposing a value-matching condition at \( \pi_F \) (for further explanation see Harrison, 1985). Since there is no optimality on the part of the leader there is
no corresponding smooth-pasting condition in this case. Solving this problem the leader’s value function is derived

\[ V_L(\pi) = \begin{cases} 
\frac{\pi}{r + h - \mu} - B_L \pi^{\beta_1} - K & \text{for } \pi < \pi_F \\
NPV(\pi) - K & \text{for } \pi \geq \pi_F 
\end{cases} \]  

(13)

where \( B_L = \frac{\pi^{1-\beta_1}}{(r+h-\mu)(r+2h-\mu)} > 0 \).

5.3. Solving the game

Without the ability to precommit to trigger points at the start of the game (for example, in contrast to the precommitment strategies used by Reinganum, 1981) the leader’s stopping point \( \pi_L \) cannot be derived as the solution to a single-agent optimization problem. Whether a firm becomes a leader, and if so the trigger point at which it invests, is determined by each firm’s incentive to pre-empt its rival and the point at which it is necessary to invest to prevent itself from being pre-empted.

As in Fudenberg and Tirole (1985) the form of the non-cooperative equilibrium depends on the relative magnitudes of the leader’s value, \( V_L \), and the value when both firms delay until the optimal joint-investment point, \( V_C \). Depending upon whether or not these functions intersect anywhere in the interval \((0, \pi_F)\) two investment patterns arise.\(^{11}\) If \( V_L \) ever exceeds \( V_C \) pre-emption incentives are too strong for joint investment to be an equilibrium and the only possible outcome is a leader–follower equilibrium in which one firm invests strictly earlier than its rival and both invest strictly prior to the optimal joint-investment time. If, on the other hand, \( V_L \) never exceeds \( V_C \) a joint-investment outcome may be sustained, although the leader–follower outcome is also an equilibrium in this case.

At the leader’s investment point, \( \pi_L \), the expected payoffs of the two firms must be equal. This follows Fudenberg and Tirole’s rent equalization principle: if it were not the case, one firm would have an incentive to deviate and the proposed outcome could not be an equilibrium. By investing earlier than its rival the leader gains the advantage of a temporary monopoly in research and has a greater likelihood of making the discovery; however, the expected value of the prize it stands to win is lower than for the follower. Hence, viewed from the start of the game, there is a trade-off between the probability of being first to make the discovery and the likely value of the prize that is gained. At \( \pi_L \) the two effects are in balance and the firms’ expected payoffs are equal. Thus, in contrast with many other games where asymmetric equilibria arise (such as Reinganum, 1981), the agents in this model are indifferent between the two roles.

Before formally describing the equilibria we must first define, and demonstrate the existence of, the leader’s trigger point, \( \pi_L \). From the rent equalization principle described above, it follows directly that \( V_L(\pi_L) = V_F(\pi_L) \). Using this equality an implicit expression for \( \pi_L \) can be derived; this is given by expression (A4.1) in the appendix when evaluated at zero. Thus it is necessary to prove the existence of a root of this expression other than, and strictly below, \( \pi_F \).

**Lemma 4.** There exists a unique point \( \pi_L \in (0, \pi_F) \) such that

\[ V_L(\pi) < V_F(\pi) \quad \text{for } \pi < \pi_L \]

\[ V_L(\pi) = V_F(\pi) \quad \text{for } \pi = \pi_L \]

\[ V_L(\pi) > V_F(\pi) \quad \text{for } \pi \in (\pi_L, \pi_F) \]

\(^{11}\) Since \( V_L(\pi) = NPV(\pi) - K \) for \( \pi \geq \pi_F \) while \( V_C(\pi) > NPV(\pi) - K \) for \( \pi < \pi_C \), the functions cannot intersect anywhere in the interval \([\pi_F, \pi_C]\).
Proposition 3 (Case 1). If $\exists \pi \in (0, \pi_F)$ such that $V_L(\pi) > V_C(\pi)$, then there exist two asymmetric leader–follower equilibria differing only in the identities of the two firms. In one equilibrium firm 1 (the leader) invests when $\pi_L$ is first reached with firm 2 (the follower) investing strictly later at $\pi_F > \pi_L$; the other equilibrium is identical except that the firms’ identities are reversed.

Proof. Case 1 is illustrated with reference to Figure 1. As $\pi$ rises from $\pi_0$ we know from the premise that a point (labelled A) will be reached where $V_L$ first exceeds $V_C$ and each firm has a unilateral incentive to deviate from the continuation strategy to become the leader. But if one firm were to pre-empt at $A$ its payoff would be strictly greater than its rival’s, since $V_L > V_F$ at this point. Lemma 4 tells us that the leader’s payoff is strictly greater the follower’s everywhere in the interval $(\pi_L, \pi_F)$, thus pre-emption rules out any putative trigger point in this range. We know also that $V_L < V_F$ for all $\pi < \pi_L$, thus prior to $\pi_L$ each firm prefers to let its rival take the lead. Furthermore, Lemma 4 tells us that $\pi_L$ is unique. After the leader has invested the follower faces the optimisation problem of Section 5.1. Hence there exists a unique equilibrium configuration in which one firm (the leader) invests when $\pi_L$ is first reached and the other (the follower) invests strictly later at $\pi_F$. Since the firms’ identities are interchangeable there are two equilibria of this type. \hfill ||

We next consider the alternative case where $V_C$ always exceeds $V_L$ and a joint-investment equilibrium is sustainable. At $\pi_C$ it is a dominant strategy to invest even though the rival will follow at once, thus there can be no equilibrium trigger point above $\pi_C$. Before describing the set of joint-investment equilibria we must first define $\pi_S$, the lowest joint-investment point such
that there is no unilateral incentive to deviate.\textsuperscript{12}

\[ \pi_S = \inf[\pi_J \in (0, \pi_C] : V_J(\pi; \pi_J) \geq V_L(\pi) \forall \pi \in (0, \pi_J)] \]  \hspace{1cm} (14)

where \( V_J(\pi; \pi_J) \) is the firm's pre-investment value function when both invest jointly, but not necessarily optimally, at an arbitrary point \( \pi_J \). This function, derived from value-matching at \( \pi_J \) and defined over the range \( (0, \pi_J] \), is given by

\[ V_J(\pi; \pi_J) = B_J \pi - \frac{hr}{r+2h-\mu} - K \]  \hspace{1cm} (15)

where \( B_J = \pi_J^{\beta_0} \).

Lemma 5. \hspace{1cm} (a) \( \pi_S \) exists and is unique whenever \( V_C(\pi) \geq V_L(\pi) \forall \pi \in (0, \pi_C) \).

(b) \( [\pi_S, \pi_C] \) forms a connected set such that \( V_J(\pi; \pi_J) \geq V_L(\pi) \forall \pi \in (0, \pi_J), \pi_J \in [\pi_S, \pi_C] \).

Proof. \hspace{1cm} See appendix. ||

Proposition 4 (Case 2). If \( V_C(\pi) \geq V_L(\pi) \forall \pi \in (0, \pi_C) \), two types of equilibria exist. The first is the leader–follower equilibrium described in Proposition 3; two equilibria of this type exist as before. The second is a joint-investment equilibrium in which both firms invest at the same trigger point \( \pi_J \in [\pi_S, \pi_C] \); there is a continuum of equilibrium trigger points over this interval.

Proof (Case 2 is illustrated with reference to Figure 2). As before, in the interval \((\pi_L, \pi_F)\) over which \( V_L > V_F \) fear of pre-emption entails that the asymmetric leader–follower outcome is an equilibrium configuration. From the premise, however, there is no unilateral incentive to deviate from the continuation strategy anywhere in the interval \((0, \pi_C)\). For \( \pi \geq \pi_C \) it is a dominant

\textsuperscript{12}. Note that the critical value \( \pi_S \) does not necessarily exist; this depends upon the relative positions of \( V_L \) and \( V_C \).
strategy to invest even though the rival will follow at once. Thus the joint-investment outcome in which both firms invest at \( \pi_C \) is also an equilibrium. From Lemma 5 any joint-investment point \( \pi_J \in [\pi_S, \pi_C] \) has the property that no unilateral deviation is profitable and is therefore an equilibrium.  

Fudenberg and Tirole (1985) argue that if one equilibrium Pareto-dominates all others it is the most reasonable outcome to expect. Using the Pareto criterion the multiplicity of equilibria described in Proposition 4 can be reduced to a unique outcome.

**Proposition 5.** Using the Pareto criterion, the multiplicity of equilibria arising in case 2 can be reduced to a unique outcome, the joint-investment equilibrium in which both firms invest when \( \pi_C \) is first reached.

**Proof.** The proof consists of two parts.

(i) All joint-investment equilibria, where these exist, Pareto-dominate the asymmetric leader–follower equilibria. From the definition of \( \pi_S \) any joint-investment trigger \( \pi_J \in [\pi_S, \pi_C] \) has the property that no unilateral deviation is profitable. Thus \( V_J(\pi) \geq V_L(\pi) \forall \pi \in (0, \pi_J] \) and the value of continuation is at least as great as the amount that a firm would gain from pre-emption at any point. Furthermore, in the leader–follower equilibrium the payoffs of both firms are strictly lower than the maximum amount obtainable, since the optimal pre-emption strategy is not an equilibrium of the non-cooperative game.

(ii) From the derivation of \( \pi_C \), it follows that the joint-investment equilibria are Pareto-ranked by their respective trigger points with trigger points closer to \( \pi_C \) Pareto-dominating all lower ones.

The asymmetric equilibria arising in case 2 are situations where the leader pre-empts purely due to the fear that its rival will do so. Such instances of ‘attack as a means of defence’ are somewhat irrational as both firms achieve higher payoffs by coordinating on any one of the symmetric equilibria. Comparing the non-cooperative trigger points with the cooperative solution a number of comparisons can be drawn. Lemma 6 compares first investment points in the non-cooperative and cooperative solutions; the full set of rankings is then given in Proposition 6.

**Lemma 6.** \( \pi_L < \pi_1 \) for \( h > 0 \).

**Proof.** See appendix.

**Proposition 6.** Trigger points in the various cases are ranked as follows:

(a) \( \pi_L < \pi_1 < \pi_C < \pi_2 \);
(b) \( \pi_L < \pi_F < \pi_C < \pi_2 \);
(c) the ranking of \( \pi_1 \) and \( \pi_F \) is ambiguous.

Whether equilibrium in any particular case involves pre-emptive or simultaneous investment depends on parameter values and can be determined numerically as follows. There is a unilateral incentive to deviate from joint investment if and only if a designated leader (which, unlike the firms in this model, can choose its investment point optimally secure in the knowledge that it cannot be pre-empted) would choose to adopt the leadership role. The investment point of the
designated leader and its option value $B_D \pi^R_0$ are derived using value-matching and smooth-pasting conditions. This problem has no closed-form solution; implicit expressions are presented in the appendix. Once a solution for $B_D$ has been obtained the equilibrium investment pattern can be determined by comparing this value with $B_C$ (defined following expression (10) above). If $B_D > B_C$ case 1 holds and the outcome is a pre-emptive leader–follower equilibrium. If, on the other hand, $B_C \geq B_D$ case 2 obtains and the Pareto dominant equilibrium entails joint investment at $\pi_C$. Some numerical results are discussed in Section 6.

6. DISCUSSION

By comparing the cooperative and non-cooperative solutions the inefficiency of non-cooperative behaviour can readily be seen. When non-cooperative behaviour gives rise to a leader–follower equilibrium pre-emption and business-stealing incentives prevent the option to invest from being held for long enough and both firms invest too soon. Although the leader gains the first-mover advantage of a temporary monopoly in research, this is subsequently undermined by the follower’s investment. The firms’ payoffs are equal, and are low compared with the other outcomes.

The joint-investment equilibrium, if achievable, is more favourable for both firms. It is identical to the outcome that would be achieved if the firms agreed to adopt a common investment rule and chose this optimally. Although it is not the cooperative optimum—as Section 4 has shown, simultaneous investment is dominated by the optimal sequential investment pattern—it could be seen as the best achievable cartel given the difficulty in agreeing asymmetric investment rules and the need for side-payments implicit in the cooperative outcome.

Interestingly, when equilibrium involves simultaneous investment the effect of non-cooperative behaviour is to increase the time to first investment: the non-cooperative trigger $\pi_C$ exceeds the first trigger point in the cooperative plan $\pi_1$. This result contrasts strongly with the usual presumption that competition undermines the ability to delay. Investment occurs too late due to the strategic behaviour of the firms who delay their investment in the fear of setting off a patent race. Hence, in this case, delay is due to strategic interactions between firms, not just the usual option effect of uncertainty. Investment is also more delayed than when a single firm holds the opportunity to invest. When investment does occur, however, a burst of research activity is seen which is excessive—under the cooperative plan the second investment would be delayed until a later date.

The type of equilibrium that emerges in any particular case depends on the balance between two opposing forces, the option value of delay and the expected benefit of pre-emption. The simultaneous investment equilibrium becomes more prevalent as the option value of delay is increased or the pre-emptive effect of earlier investment is reduced. Numerical analysis indicates that simultaneous investment becomes the equilibrium outcome as, ceteris paribus, volatility $\sigma$ rises, the hazard rate $h$ falls, or the pure discount rate $r$ increases. (As for financial options, an increase in pure discounting reduces the current value of the investment cost, or strike price, paid at some date in the future, raising option values.)

These findings have a number of implications for the understanding and assessment of empirical investment behaviour. Since strategic interactions, as well as uncertainty, have significant effects on the timing and pattern of investment, empirical studies of investment may be improved by including measures of industry concentration and strategic advantages as explanatory variables. If pre-emption effects are strong competition tends to speed up investment,

13. With $K$ adjusted appropriately so that the project’s expected value remains constant.
14. With $\mu$ adjusted in line so that the opportunity cost $\delta = r - \mu$ remains constant.
which then takes place sequentially as firms avoid competing head-to-head. Greater volatility, on the other hand, increases the likelihood that a patent race will occur at a later date, with a sudden burst of competitive activity ending a prolonged period of stagnation—a phenomenon similar to that described by Choi (1991) but arising for different reasons.

Some welfare implications can also be drawn from this analysis. Although a full welfare assessment requires a value function for consumers to be specified so that the social optimum can be determined, conclusions can be drawn for one simple case. If the consumer surplus arising from the innovation remains in fixed proportion to π as this varies over time (i.e. the patent-holder extracts the same proportion of the social surplus from the innovation at all times), the social optimum coincides with the cooperative solution. The social planner would phase investment progressively over time, choosing the same trigger points as the cooperating firms. A sudden patent race is therefore seen to be socially inefficient. Comparing the two forms the non-cooperative equilibria, although simultaneous investment is found to be sub-optimal it is preferable to the leader–follower equilibrium in which both firms invest too soon and valuable options for the future are destroyed. Only if for some reason early investment has significant external benefits for consumers—and the mere existence of consumer surplus is not sufficient for this—would the social planner prefer the pre-emptive equilibrium.

Turning next to policy issues, the analysis has implications for the assessment of R&D joint ventures. It provides further justification for adopting a liberal approach to cooperative R&D, in addition to existing arguments concerning complementary skills, spillover effects, and the scale and riskiness of R&D investment. When the option to delay is socially as well as privately beneficial this analysis strongly supports the creation of an R&D joint venture with the freedom to choose the timing and scale of investment cooperatively. Of course, this and other benefits of cooperation must be balanced against its possible detriments, especially the weakening of efficiency incentives and the possibility that cooperation will extend downstream to product market collusion.

It is interesting to note that in the case where a joint venture would be the most desirable, namely that in which a pre-emptive leader–follower equilibrium would otherwise occur, the joint venture would choose to delay R&D investment. This is in stark contrast with the usual policy prescription whereby firms are required to demonstrate that the joint venture will invest in projects that would not otherwise be undertaken (at the present time). A significant change in approach on the part of competition authorities might be required to take account of this point! When non-cooperative equilibrium takes the simultaneous investment form, however, no such conflict arises: the joint venture will undertake the first investment earlier than would otherwise be the case, and further investment will be phased in at a later date as and when this becomes optimal.

7. CONCLUDING REMARKS

This paper has shown that, in contrast to initial expectations, competition between a small number of firms does not necessarily undermine the option to delay. Instead the fear of sparking a patent race may internalize the effect of competition, further raising the value of delay and increasing the time before any investment takes place. A number of implications for empirical and policy analysis follow from the analysis.

15. Given that this proportion is determined largely by the duration of the patent and the degree of monopoly power conveyed by the patent grant, this would not seem to be an unreasonable assumption.

16. All values and trigger points are scaled up by the same proportion, leaving the optimal timing of investment unchanged.
From the results are robust to changes in the precise structure of the model. Although GBM is a convenient and tractable form, alternative stochastic processes, such as ones exhibiting mean-reversion or intermittent jumps, would generate similar qualitative results. More sophisticated research technologies could also be incorporated. For example, the hazard rate may increase with cumulative R&D spending as a result of learning-by-doing. Note that in this case the leader has a permanent rather than a temporary advantage, strengthening pre-emption incentives. Alternatively, if the probability of discovery is not known a priori and the hazard rate is therefore an expectation, updating from fruitless research experience will cause the hazard rate to fall over time.

The model could be extended in a number of ways. This paper has focused on the symmetric two-firm case. If the firms’ research technologies are instead allowed to differ such that one is more efficient, the identities of the leader and follower will be uniquely defined and the more efficient firm will receive a strictly greater expected payoff. An increase in the number of firms, however, is more problematic. As explained by Fudenberg and Tirole (1985, Section 5), rent equalization holds only in the two-firm case; with three or more symmetric firms equilibrium behaviour is more complicated and asymmetric payoffs are possible. The impact of rival investment in research may also have more complicated effects than those considered in this model. Congestion effects, such as a shortage of skilled workers, may reduce the efficiency of research as more firms invest, raising the advantage of earlier investment. Informational spillovers between firms, on the other hand, would cause a firm’s hazard rate to rise when its rival invests. This generates an additional motive for delay, as a firm gains by free-riding on the research efforts of its rival.

APPENDIX

Proof of Lemma 1. From (8) we can write

\[ Y(\pi) = (\rho_0 - 1) \frac{h\pi}{(r + h - \mu)} - \frac{(\beta_1 - \rho_0)}{(\beta_1 - 1)} K \left( \frac{\pi}{\pi_2} \right)^{\beta_1} - \rho_0 K. \]  

(A1.1)

\( \pi_1 \) is the root of this function which lies in the interval \([0, \pi_2]\). To demonstrate the existence and uniqueness of such a root, and the fact that \( \pi_1 \in (0, \pi_2) \), it is sufficient to show that the continuous function \( Y(\pi) \) has the following properties:

(i) \( Y''(\pi) = -\beta_1(\beta_1 - \rho_0)K\pi_2^{-\beta_1}\pi^{\beta_1 - 2} < 0 \) for \( \pi > 0 \), thus \( Y(\pi) \) is strictly concave over \((0, \infty)\);

(ii) \( Y(0) = -\rho_0 K < 0 \);

(iii) \( Y(\pi_2) > 0 \) for \( h > 0 \). Substituting from (6) for \( \pi_2 \) we can write

\[ Y(\pi_2) = \frac{KH}{(\beta_1 - 1)(r - \mu)}. \]  

(A1.2)

where \( H = 2h\beta_1(\rho_0 - 1) - (r - \mu)(\beta_1 - \rho_0) \). A necessary and sufficient condition for \( Y(\pi_2) > 0 \) is \( H > 0 \) \( \forall h > 0 \). When \( h = 0, \beta_1 = \rho_0 \) and \( H = 0 \). Evaluating the first derivative we find

\[ \frac{\partial H}{\partial h} = 2\beta_1(\rho_0 - 1) + \frac{4h(\beta_0 - 1) - 2(r - \mu)}{\sigma^2(2\beta_1 - 1) + 2\mu}. \]  

(A1.3)

Multiplying by the denominator on the RHS (which is strictly positive) we can write

\[ L = [\sigma^2(2\beta_1 - 1) + 2\mu] \frac{\partial H}{\partial h} \]

\[ = 2\sigma^2\beta_1(\rho_0 - 1)(2\beta_1 - 1) + 4\mu\beta_1(\rho_0 - 1) + 4h(\beta_0 - 1) - 2(r - \mu). \]

A necessary and sufficient condition for \( \frac{\partial H}{\partial h} > 0 \) is \( L > 0 \).

Recalling that \( \beta_1 \) is increasing in \( h \), it is clear that \( L \) increases with \( h \) and it is sufficient to demonstrate that \( L > 0 \) when \( h = 0 \). We can derive

\[ L|_{h=0} = 4\beta_0(\rho_0 - 1) + 4\beta_0(r - \mu\rho_0) - 2(r - \mu). \]
Recalling that \( 1 < \beta_0 < r/\mu \) the upper and lower limits can be derived

\[
0 < 2(r - \mu) < L|h=0 < 2(r - \mu) \left( \frac{2r}{\mu} - 1 \right). \tag{A1.4}
\]

**Proof of Lemma 2.** The objective is to compare expressions (6) and (9). This reduces to a comparison between

\[
\frac{\beta_0}{(\beta_0 - 1)} \quad \tag{A2.1}
\]

and

\[
\frac{\beta_1}{(\beta_1 - 1)} \left( \frac{r + h - \mu}{r - \mu} \right). \tag{A2.2}
\]

Recall that \( \beta_0 \) is independent of \( h \) while \( \beta_1 \) is increasing in \( h \). When \( h = 0 \), \( \beta_1 = \beta_0 \) and the two expressions are identical. Expressing (A2.2) in the form \( M(h)/N(h) \), where \( M(h) = \beta_1(r + h - \mu) \) and \( N(h) = (\beta_1 - 1)(r - \mu) \), differentiation with respect to \( h \) yields

\[
\frac{\partial M}{\partial h} = \frac{\partial N}{\partial h} + h\frac{\partial \beta_1}{\partial h} + \beta_1 > \frac{\partial N}{\partial h}. \tag{A2.3}
\]

Thus, (A2.2) is strictly increasing in \( h \) and \( \pi_2 > \pi_C \) for \( h > 0 \).

**Proof of Lemma 3.** From Lemma 1, to show that \( \pi_C > \pi_1 \) it is necessary and sufficient to demonstrate that \( Y(\pi_C) > 0 \) (given that it is already known from Lemma 2 that \( \pi_C < \pi_2 \)). Substituting for \( \pi_C \) we can write

\[
Y(\pi_C) = \frac{K\beta_0 h}{(r + h - \mu)} - \frac{K(\beta_1 - \beta_0)}{(\beta_1 - 1)} \left[ \frac{\beta_0}{(\beta_0 - 1)} \left( \frac{r - \mu}{\beta_1} \right) (r - h - \mu) \right] \beta_1. \tag{A3.1}
\]

As a corollary of Lemma 2 we know that the term in square brackets is less than unity (as this is \( \pi_C/\pi_2 \)). Since \( \beta_1 > 1 \), we know that

\[
Y(\pi_C) > \frac{K\beta_0 h}{(r + h - \mu)} - \frac{K(\beta_1 - \beta_0)}{(\beta_1 - 1)} \left[ \frac{\beta_0}{(\beta_0 - 1)} \left( \frac{r - \mu}{\beta_1} \right) (r - h - \mu) \right] = \frac{K\beta_0}{(r + h - \mu)} Z(h)\]

where

\[
Z(h) = h - \frac{(r - \mu)}{(\beta_0 - 1)} \left[ 1 - \frac{\beta_0}{\beta_1} \right]. \tag{A3.2}
\]

Thus, to prove Lemma 3 it is sufficient to show that \( Z(h) > 0 \forall h > 0 \). This follows from the following facts:

(i) \( Z(0) = 0 \);
(ii) \( Z(h) \) is strictly convex;
(iii) \( Z'(h) \) evaluated at \( h = 0 \) is strictly positive.

(i) is straightforward. To demonstrate (ii) and (iii) we start by taking partial derivatives with respect to \( h \)

\[
\frac{\partial Z}{\partial h} = 1 - \frac{\beta_0}{(\beta_0 - 1)} \left( \frac{r - \mu}{\beta_1} \right) \left( \frac{2}{(2\beta_1 - 1)\sigma^2 + 2\mu} \right) \tag{A3.3}
\]

and

\[
\frac{\partial^2 Z}{\partial h^2} = \frac{4(r - \mu)}{\beta_1^3} \left[ \frac{\beta_0}{(\beta_0 - 1)} \left( \frac{3\beta_1 - 1}{(2\beta_1 - 1)\sigma^2 + 2\mu} \right) \right] > 0. \tag{A3.4}
\]

Recalling that \( \beta_1 = \beta_0 \) when \( h = 0 \), after some manipulation we can write

\[
\frac{\partial Z}{\partial h} \bigg|_{h=0} = 1 - \frac{(r - \mu)}{(r - \mu) + (\beta_0 - 1)G(r)} \tag{A3.5}
\]

where \( G(r) = 2r - (\beta_0 + 1)\mu \). As \( r \to \mu, \beta_0 \to 1 \) and so \( G(r) \to 0 \). Taking partial derivatives we can write

\[
\frac{\partial G}{\partial r} = 2 - \frac{2\mu}{2\mu + (2\beta_0 - 1)\sigma^2} > 0. \tag{A3.6}
\]
Thus \( G(r) > 0 \) \( \forall r > \mu \) and \( Z'(h) \) evaluated at \( h = 0 \) is strictly positive. Hence \( Z(h) > 0 \) \( \forall h > 0 \), which is sufficient to demonstrate that \( P(\pi C) > 0 \). Thus, \( \pi C > \pi 1 \) for \( h > 0 \). \|

**Proof of Lemma 4.** We start by defining the function \( P(\pi) = V_L(\pi) - V_F(\pi) \) describing the gain to pre-empting one’s opponent as opposed to being pre-empted. Expanding using equations (12) and (13) we can write

\[
P(\pi) = \frac{h \pi}{r + h - \mu} - K - \left( \frac{\pi}{\pi F} \right) \frac{h \pi F}{r + h - \mu} - K \] (A4.1)

for \( \pi \in (0, \pi F) \).

The following steps are sufficient to demonstrate the existence of a root in the interval \( (0, \pi F) \).

(i) Evaluating \( P(\pi) \) at zero yields \( \pi 0 = -K < 0 \).

(ii) Evaluating \( P(\pi) \) at \( \pi F \) yields \( \pi F = 0 \).

(iii) Evaluating the derivative \( P'(\pi) \) at \( \pi F \) it can be shown that

\[
\text{sgn} \left\{ \frac{dP}{d\pi} \right\}_{\pi F} = \text{sgn} \left\{ -\frac{h}{(r + h - \mu)} (r + h) (r + 2h) K \right\} < 0. \] (A4.2)

Thus, \( P(\pi) \) must have at least one root in the interval \( (0, \pi F) \).

Uniqueness of the root \( \pi L \) and the validity of the two inequalities can be proven by demonstrating strict concavity of \( P(\pi) \) over \( (0, \pi F) \). By differentiation we can derive

\[
P''(\pi) = -\beta_1 (\beta_1 - 1) \pi F - \beta_1 \left( \frac{h \pi F}{r + h - \mu} - K \right) \pi F - 2 < 0 \text{ for } \pi > 0. \] (A4.3)

Thus the root is unique, with \( P(\pi) < 0 \) for \( \pi \in (0, \pi L) \) and \( P(\pi) > 0 \) for \( \pi \in (\pi L, \pi F) \).

The final equality is demonstrated by considering the follower’s optimal behaviour over the range \([\pi F, \infty)\). This interval is the follower’s stopping region over which its best response to investment by the leader is to invest at once. Thus, the values of the leader and follower are equal over this range. \|

**Proof of Lemma 5.**

**Proof.**

(a) To demonstrate existence we start by showing that \( V_j(\pi; \pi C) = V_C(\pi) \). With some simplification, the expressions for \( V_j \) and \( \pi C \) yield

\[
V_j(\pi; \pi C) = \frac{h \pi C^{1-\beta_0}}{r + 2h - \mu} \pi C_0^{\beta_0} = B C \pi C_0^{\beta_0} = V_C(\pi). \] (A5.1)

It then follows from the premise that there exists at least one \( \pi j \in (0, \pi C) \) such that \( V_j(\pi; \pi j) \geq V_j(\pi) \forall \pi \in (0, \pi j) \) at the very least \( \pi C \) itself satisfies this condition. \( \pi S \) is then defined to be the smallest element of the set of joint-investment points satisfying the condition.

With \( \pi S \) defined as the lowest joint-investment point such that the two functions \( V_L(\pi) \) and \( V_j(\pi; \pi S) \) just touch another, a sufficient condition for uniqueness of \( \pi S \) is that \( V_j(\pi; \pi j) \) is strictly increasing in \( \pi j \) for \( \pi j \in (0, \pi C) \). We derive

\[
\frac{\partial V_j}{\partial \pi j} = \left[ \beta_0 K - (\beta_0 - 1) \frac{h \pi j}{h + 2r - \mu} \right] \pi j^{-(\beta_0 + 1)} \pi j \beta_0 > 0 \text{ for } \pi j \in (0, \pi C). \] (A5.2)

Thus, for any \( \pi j < \pi C \) a higher value of \( \pi j \) entails a strictly higher value of \( V_j \) at any given value of \( \pi \).

(b) To show that \( [\pi S, \pi C] \) forms a connected set satisfying the condition that \( V_j(\pi; \pi j) \geq V_L(\pi) \forall \pi \in (0, \pi j), \pi j \in [\pi S, \pi C] \) it is sufficient to show that \( V_j(\pi; \pi j) \) is increasing in both of its arguments for all \( \pi j \in (0, \pi C) \). Since \( \partial V_j/\partial \pi j \) evaluated at \( \pi C \) equals zero, we can write

\[
\frac{\partial V_j}{\partial \pi j} \geq 0 \text{ for } \pi j \in (0, \pi C). \] (A5.3)

It can easily be seen that

\[
\frac{\partial V_j}{\partial \pi} = B j(\pi) \beta_0 \pi^{\beta_0 - 1} > 0 \quad \forall \pi > 0. \] (A5.4)
The designated leader's investment problem

The designated leader's investment point, \( \pi_D \), is defined implicitly by

\[
(\beta_1 - \beta_0) B_L \pi_D^{\beta_1} + (\beta_0 - 1) \frac{h \pi_D}{r + h - \mu} - \beta_0 K = 0
\]

where \( B_L \) is as defined following expression (13). Having solved numerically for the value of the trigger point \( \pi_D \), the option value constant \( B_D \) is then defined by

\[
B_D = \frac{\pi_D^{\beta_0}}{\beta_0} \left( \frac{h \pi_D}{r + h - \mu} - \beta_1 B_L \pi_D^{\beta_1} \right).
\]

Proof of Lemma 6.

Proof. From rent equalization at \( \pi_L \) we know that \( V_L(\pi_L) = V_F(\pi_L) \). Using (12) and (13) to substitute for the respective value functions at this point, and their derivatives, we can write

\[
V'_F(\pi_L) = V'_L(\pi_L) - \frac{\beta_1 K}{\pi_L} + \frac{(\beta_1 - 1) h}{(r + h - \mu)}.
\]

(A6.1)

From Lemma 4 we know that \( V_L \) crosses \( V_F \) at \( \pi_L \) from below and must therefore have the steeper slope, thus \( V'_L(\pi_L) > V'_F(\pi_L) \). Thus, the following upper bound for \( \pi_L \) can be derived:

\[
\pi_L < \frac{\beta_1}{(\beta_1 - 1)} \frac{(r + h - \mu)}{h} K \equiv \pi_M.
\]

(A6.2)

Given the shape of the function \( Y(\pi) \) (defined by (A3.1) above) of which \( \pi_1 \) is the root, and since \( \pi_M < \pi_2 \), it is sufficient to show that \( Y(\pi_M) < 0 \). It can readily be shown that

\[
Y(\pi_M) = \frac{K}{(\beta_1 - 1)} \left( \beta_0 - \beta_1 - (\beta_1 - \beta_0) \left[ \frac{(r - \mu)}{r + 2h - \mu} \right] \right) < 0 \quad \text{for } h > 0.
\]

(A6.3)

Thus, \( \pi_L < \pi_M < \pi_1 \).

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