Second Order Stochastic Dominance

Definition
Suppose the random variables $X$ and $Y$ have support on $[l, u]$. Then $X$ second-order stochastically dominates $Y$ if
\[ \int_l^a \Pr[X > t]dt \geq \int_l^a \Pr[Y > t]dt \]
for all $a$.

Results
1. If $X$ first order stochastically dominates $Y$, then $X$ second-order stochastically dominates $Y$.
   \[ \Pr[X > t] \geq \Pr[Y > t] \]
   for all $t$. Integrating both sides over $t$ gets the desired result.
2. If $X$ second-order stochastically dominates $Y$, then $E[X] \geq E[Y]$.
   \[ E[X] = l + \int_l^u [1 - F(t)]dt \]
   By second order stochastic dominance, this is greater than
   \[ E[Y] = l + \int_l^u [1 - G(t)]dt \]
3. If $X$ second-order stochastically dominates $Y$, then $h(X)$ second-order stochastically dominates $h(Y)$ for any increasing and concave function $h$.
   \[ \Pr[X' > x] = \int_{l'}^{a'} \Pr[X > h^{-1}(x)]dx = \int_{l'}^{a'} [1 - F(h^{-1}(x))]dx \]
   Let $t = h^{-1}(x)$. Then $h(t) = x$. Hence, $h'(t)dt = dx$. Using the method of substitution, the integral is equal to
   \[ \int_l^a (1 - F(t))h'(t)dt \]
Let $F^*(a) = \int_1^a (1 - F(t))dt$. Using integration by parts, this is equal to

$$\left[ h'(t)F^*(t) \right]_{t=1}^{t=a} - \int_1^a h''(t)F^*(t)dt$$

Since $F^*(1) = 0$, we have

$$\int_1^a \Pr[X' > x]dx = h'(a)F^*(a) - \int_1^a h''(t)F^*(t)dt.$$ 

By second-order stochastic dominance, $F^*(t) \geq G^*(t)$ for all $t$. Furthermore, since $h'(a) \geq 0$ and $h''(t) \leq 0$, the above is greater than

$$\int_1^a \Pr[Y' > y]dy = h'(a)G^*(a) - \int_1^a h''(t)G^*(t)dt$$

for all $a'$.

4. $X$ second-order stochastically dominates $Y$ if and only if

$$E[h(X)] \geq E[h(Y)]$$

for all increasing and concave function $h$.

Proof. By 2 and 3 above, we obtain $\text{XSOSD } Y \implies E[h(X)] \geq E[h(Y)]$.

To prove the converse, suppose $E[h(X)] \geq E[h(Y)]$ for all increasing and concave function $h$. Consider the function,

$$h(t) = \begin{cases} 
  t & \text{if } t \leq a, \\
  a & \text{if } t > a.
\end{cases}$$

Obviously $h$ is increasing and concave. Now,

$$E[h(X)] = \int_1^a tf(t)dt + \int_a^1 af(t)dt$$

$$= \left[-t(1 - F(t))\right]_{t=1}^{t=a} + \int_1^a [1 - F(t)]dt + a(1 - F(a))$$

$$= \int_1^a [1 - F(t)]dt + l.$$ 

Similarly,

$$E[h(Y)] = \int_1^a [1 - G(t)]dt + l.$$ 

Therefore $E[h(X)] \geq E[h(Y)]$ implies that $X$ second-order stochastically dominates $Y$. 

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4. If $X$ second-order stochastically dominates $Y$, and if $X$ and $Y$ have the same mean, then $E[h(X)] \geq E[h(Y)]$ for all concave function $h$ (notice that $h$ does not have to be increasing).

**Proof.** Using integration by parts twice, we have

$$E[h(X)] = \int_l^u h(t)f(t)dt$$

$$= [h(t)(1 - F(t))]_{t=l}^{t=u} + \int_l^u h'(t)(1 - F(t)) dt$$

$$= [h(t)(1 - F(t))]_{t=l}^{t=u} + [h'(t)F^*(t)]_{t=l}^{t=u} + \int_l^u h''(t)F^*(t) dt$$

$$= h(l) + h'(u)F^*(u) + \int_l^u h''(t)F^*(t) dt.$$  

Similarly,

$$E[h(Y)] = h(l) + h'(u)G^*(u) - \int_l^u h''(t)G^*(t) dt$$

But since $F^*(u) = G^*(u)$ if $X$ and $Y$ have the same mean, and since $F^*(t) \geq G^*(t)$, we have

$$E[h(X)] - E[h(Y)] = \int_l^u h''(t)(G^*(t) - F^*(t)) dt \geq 0.$$  

5. If $X$ second-order stochastically dominates $Y$, and if $X$ and $Y$ have the same mean, then $X$ has a smaller variance than $Y$.

**Proof.**


Since $X^2$ is convex, result 4 implies that $E[X^2] \leq E[Y^2]$.  

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