Random Assignment Mechanisms in Large Markets

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Based on two joint papers:

"Incentives in the Probabilistic Serial Mechanism"
with Mihai MANEA

"Asymptotic Equivalence of Probabilistic Serial and Random Priority Mechanisms"
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INCENTIVES IN THE PROBABILISTIC SERIAL MECHANISM

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Abstract. The probabilistic serial mechanism (Bogomolnaia and Moulin 2001) is ordinally efficient but not strategy-proof. We study incentives in the probabilistic serial mechanism for large allocation problems. We establish that, for a fixed set of object types and an agent with a given expected utility function, if there are sufficiently many copies of each object type, then reporting ordinal preferences truthfully is a weakly dominant strategy for the agent (regardless of the number of other agents and their preferences). The non-manipulability and the ordinal efficiency of the probabilistic serial mechanism support its implementation instead of random serial dictatorship in large assignment problems. *JEL Classification Numbers*: C70, D61, D63.

Keywords: random assignment, probabilistic serial mechanism, ordinal efficiency, exact strategy-proofness in large markets, random serial dictatorship.

1. Introduction

In an assignment problem a set of indivisible objects that are collectively owned must be allocated to a number of agents, who can each consume at most one object. University house allocation and student placement in public schools are examples of important assignment problems.[1] The allocation mechanism needs to be fair and efficient. In many applications monetary transfers are precluded and fairness concerns motivate random

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[1]See Abdulkadiroğlu and Sönmez (1999) and Chen and Sönmez (2002) for house allocation, and Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003b) for student placement. Practical considerations in designing student placement mechanisms in New York City and Boston are discussed by Abdulkadiroğlu, Pathak, and Roth (2005) and Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005).

assignments. Often the allocation must be based on the agents' reports of ordinal preferences over objects rather than cardinal preferences, as elicitation of cardinal preferences may prove difficult.[2]

There are two important solutions to the random assignment problem: the random serial dictatorship mechanism (Abdulkadiroğlu and Sönmez 1998) and the probabilistic serial mechanism (Bogomolnaia and Moulin 2001). Random serial dictatorship draws each possible ordering of the agents randomly (with equal probability) and, for each realization of the ordering, assigns the first agent his most preferred object, the next agent his most preferred object among the remaining ones, and so on. This mechanism is strategy-proof and ex post efficient. Random serial dictatorship is used for house allocation in universities and for student placement in public schools.

Despite its ex post efficiency, random serial dictatorship may result in unambiguous efficiency loss ex ante. Bogomolnaia and Moulin (2001) provide an example in which the random serial dictatorship assignment is first-order stochastically dominated by another random assignment with respect to the ordinal preferences of every agent. A random assignment is called ordinally efficient if it is not first-order stochastically dominated with respect to the ordinal preferences of every agent by any other random assignment. Clearly, any ordinally efficient random assignment is ex post efficient. Ordinal efficiency is a suitable efficiency concept in the context of allocation mechanisms based solely on ordinal preferences.

Bogomolnaia and Moulin propose the probabilistic serial mechanism as an alternative to random serial dictatorship. The idea is to regard each object as a continuum of "probability shares." Each agent "eats" his most preferred available object with speed one at every point in time between 0 and 1. The probabilistic serial (random) assignment is defined as the profile of shares of objects that agents eat by time 1. The ensuing random assignment is ordinally efficient and envy-free with respect to the reported preferences.

However, the desirable properties of the probabilistic serial mechanism come at a cost. The mechanism is not strategy-proof, which means that in some circumstances an agent can obtain a more preferred random assignment (with respect to his true expected utility

[2]The market-like mechanism of Hylland and Zeckhauser (1979) is one of the few solutions proposed for the random assignment problem where agents report cardinal preferences.

function) by misstating his ordinal preferences. When agents report false preferences, the probabilistic serial assignment is not necessarily ordinally efficient or envy-free with respect to the true preferences. Whether the probabilistic serial mechanism is an appropriate solution to the random assignment problem has been unclear due to its incentive issues.

We show that agents have incentives to report their ordinal preferences truthfully in the probabilistic serial mechanism if the market is sufficiently large. More specifically, our main result is that, for a fixed set of object types and an agent with a given expected utility function over these objects, if the number of copies of each object type is sufficiently large, then truthful reporting of ordinal preferences is a weakly dominant strategy for the agent (for any set of other participating agents and their preferences). The incentive compatibility of the probabilistic serial mechanism we discover, together with its better efficiency and fairness properties, supports its use rather than the random serial dictatorship mechanism in large allocation problems.

We develop a lower bound on the supply of each object type sufficient for truth-telling to be a weakly dominant strategy for an agent. We show by example that the bound cannot be improved by a factor greater than $x \approx 1.76322$.

In our setting the large market assumption entails the existence of a large supply of each object type. This assumption is satisfied by several interesting models. For instance, the "replica economy" model often used to discuss asymptotic properties of markets is a special case of our setting (since the number of copies of each object type is large in an economy that is replicated many times). Also, the assumption is natural in applications. In the context of university housing, rooms may be divided into several categories according to building and size; all rooms in the same category are considered identical.[3] In the case of student placement in public schools, there are typically many identical seats at each school. As an illustration, consider a school choice setting where a student finds only 10 schools acceptable, and his utility difference between any two consecutively ranked schools is constant. Our main result implies that if there are at least

[3]For example, the assignment of rooms in Harvard graduate dorms is based only on preferences over eight types of rooms—there are two possible room sizes in each of four buildings.

18 seats at every school, then truth-telling is a weakly dominant strategy for the student in the probabilistic serial mechanism.

Related literature. Manea (2009) establishes that the fraction of preference profiles for which the random serial dictatorship assignment is ordinally efficient vanishes in large allocation problems. This result provides additional support to the use of the probabilistic serial mechanism. Simulations based on real preferences also suggest that the probabilistic serial mechanism achieves an efficiency gain over random serial dictatorship in large markets. Using the data of student placement in public schools in New York City, Pathak (2006) compares the resulting random allocations for each student under the two mechanisms in terms of first-order stochastic dominance. He finds that about 50% of the students are better off under the probabilistic serial mechanism, while about 6% are better off under the random serial dictatorship mechanism.[4]

Che and Kojima (2009) prove that the assignments in the probabilistic serial and random serial dictatorship mechanisms converge to the same limit as the supply of each object goes to infinity. Hence the magnitude of the efficiency loss under random serial dictatorship may diminish in large allocation problems. However, the two mechanisms are equivalent only asymptotically, and the paper does not analyze the speed of convergence to the common limit. By contrast, our paper shows incentive compatibility of the probabilistic serial mechanism in a large but finite allocation problem, and offers a lower bound on the size of the problem which is sufficient for this conclusion.

Incentive properties in large markets have been investigated in various areas of economics. For pure exchange economies, Roberts and Postlewaite (1976) show that agents have diminishing incentives to misrepresent demand functions in the competitive mechanism as the market becomes large. Similarly, in the context of double auctions, Gresik and Satterthwaite (1989), Rustichini, Satterthwaite, and Williams (1994), and Cripps and Swinkels (2006) show that equilibrium behavior converges to truth-telling as the number of traders grows. In the two-sided matching setting, Roth and Peranson (1999), Immorlica and Mahdian (2005) and Kojima and Pathak (2009) show that the deferred acceptance

[4]For the rest of the students, the random allocations corresponding to the two mechanisms are not comparable in terms of first-order stochastic dominance.

algorithm proposed by Gale and Shapley (1962) is difficult to manipulate profitably when the number of participants become large. Most of this research shows either that the gain from manipulation converges to zero or that equilibrium behavior converges to truth-telling in the limit as the market becomes large. In contrast to these "approximate" and "asymptotic" results, we show that truth-telling is an exact weakly dominant strategy in the probabilistic serial mechanism for finitely large markets.[5]

There is a growing literature on random assignment and ordinal efficiency. Abdulkadiroğlu and Sönmez (2003a) provide a characterization of ordinal efficiency based on the idea of dominated sets of assignments. McLennan (2002) proves that any random assignment which is ordinally efficient for some ordinal preferences is welfare-maximizing with respect to some expected utility functions consistent with the ordinal preferences. A short constructive proof is offered by Manea (2008). Kesten (2006) introduces the top trading cycles from equal division mechanism, and shows that it is equivalent to the probabilistic serial mechanism. The probabilistic serial mechanism is extended to cases with non-strict preferences, existing property rights, and muti-unit demands by Katta and Sethuraman (2006), Yilmaz (2006), and Kojima (2007), respectively. On the restricted domain of the scheduling problem, Crès and Moulin (2001) show that the probabilistic serial mechanism is group strategy-proof and first-order stochastically dominates the random serial dictatorship mechanism, and Bogomolnaia and Moulin (2002) find two characterizations of the probabilistic serial mechanism.

The rest of the paper is organized as follows. Section 2 describes the model. The main result is presented in Section 3, with the proof relegated to the Appendix. Section 4 provides a detailed example, and Section 5 concludes.

2. Model

A *random assignment problem* is a quadruple $\Gamma = (N, (\succ_i)_{i \in N}, \hat{O}, (q_a)_{a \in \hat{O}})$. N and $\hat{O}$ represent (finite) sets of *agents* and *proper object types*, respectively. The *quota* (number

[5] However, Jackson (1992) notes that truth-telling becomes a weakly dominant strategy in the competitive mechanism for large economies when agents are constrained to report from a finite set of demand functions. In contrast, Jackson and Manelli (1997) show that Nash equilibrium behavior in the competitive mechanism need not converge to truth-telling in large economies with unrestricted demand functions.

of copies) of object a is denoted by q_a. There is an infinite supply of a *null object* $\emptyset$ (which does not belong to $\hat{O}$), $q_\emptyset = +\infty$. Each agent $i \in N$ has a *strict preference* $\succ_i$ over $O := \hat{O} \cup \{\emptyset\}$. We write $a \succeq_i b$ if and only if $a \succ_i b$ or $a = b$. When N is fixed, $\succ$ denotes $(\succ_i)_{i \in N}$ and $\succ_{N'}$ denotes $(\succ_i)_{i \in N'}$ (for $N' \subset N$).

A *deterministic assignment* for the problem Γ is a matrix $X = (X_{ia})$, with rows indexed by $i \in N$ and columns by $a \in O$, such that $X_{ia} \in \{0, 1\}$ for all i and a, $\sum_{a \in O} X_{ia} = 1$ for all i, and $\sum_{i \in N} X_{ia} \leq q_a$ for all a. The value of X_{ia} is $1\,(0)$ if agent i receives (does not receive) object a at the assignment X. Hence the constraints $\sum_{a \in O} X_{ia} = 1$ and $\sum_{i \in N} X_{ia} \leq q_a$ mean that i receives exactly one object and at most q_a agents receive a at the assignment X.

A *lottery assignment* is a probability distribution w over the set of deterministic assignments, where $w(X)$ denotes the probability of the assignment X. A *random assignment* is a matrix $P = (P_{ia})$, with $P_{ia} \geq 0$ for all i and a, $\sum_{a \in O} P_{ia} = 1$ for all i, and $\sum_{i \in N} P_{ia} \leq q_a$ for all a; P_{ia} stands for the probability that agent i receives object a. A lottery assignment w *induces* the random assignment $\sum_X w(X)X$. The entry (i, a) in this matrix represents the probability that agent i is assigned object a under w. The following proposition is a generalization of the Birkhoff-von Neumann theorem (Birkhoff (1946) and von Neumann (1953)).

Proposition 1. *Every random assignment can be written as a convex combination of deterministic assignments.*[6]

The proof is in the Appendix. By Proposition 1, any random assignment is induced by a lottery assignment. Henceforth, we identify lottery assignments with the corresponding random assignments and use these terms interchangeably.

We assume that each agent has a *von Neumann-Morgenstern expected utility function* over random assignments. The *utility index* of agent i is a function $u_i : O \to \mathbb{R}$. We extend the domain of u_i to the set of random assignments as follows. Agent i's expected utility for the random assignment P is $u_i(P) = \sum_{a \in O} P_{ia} u_i(a)$. We say that u_i is *consistent* with $\succ_i$ when $u_i(a) > u_i(b)$ if and only if $a \succ_i b$.

[6]There may be multiple convex combinations of deterministic assignments that induce the same random assignment.

A random assignment P *ordinally dominates* another random assignment P' *at* $\succ$ if

$$\sum_{b \succeq_i a} P_{ib} \geq \sum_{b \succeq_i a} P'_{ib}, \ \forall i \in N, \forall a \in O,$$

with strict inequality for some i, a. A random assignment is *ordinally efficient at* $\succ$ if it is not ordinally dominated at $\succ$ by any other random assignment. Suppose that P ordinally dominates P' at $\succ$. Then each agent i weakly prefers P to P' in terms of first-order stochastic dominance with respect to $\succ_i$. Equivalently, every agent i weakly prefers P to P' according to any expected utility function consistent with $\succ_i$.

We extend the *probabilistic serial* mechanism proposed by Bogomolnaia and Moulin (2001) to our setting. Each object is viewed as a divisible good of "probability shares." Each agent "eats" his most preferred available object with speed one at every time $t \in [0,1]$—object a is *available* at time t if less than q_a share of a has been eaten away by t. The resulting profile of object shares that agents eat by time 1 corresponds to a random assignment, which is the *probabilistic serial (random) assignment.*

Formally, the *(symmetric simultaneous) eating algorithm* defines the probabilistic serial assignment for the preference profile $\succ$ as follows. For any $a \in O' \subset O$, let $N(a, O') = \{i \in N | a \succeq_i b, \forall b \in O'\}$ represent the set of agents whose most preferred object in O' is a. Set $O^0 = O$, $t^0 = 0$, and $P^0_{ia} = 0$ for every $i \in N$ and $a \in O$. For all $v \geq 1$, given $O^0, t^0, (P^0_{ia}), \dots, O^{v-1}, t^{v-1}, (P^{v-1}_{ia})$, define

$$t^v = \min_{a \in O^{v-1}} \max \left\{ t \in [0,1] \middle| \sum_{i \in N} P^{v-1}_{ia} + |N(a, O^{v-1})|(t - t^{v-1}) \leq q_a \right\},$$

$$O^v = O^{v-1} \setminus \left\{ a \in O^{v-1} \middle| \sum_{i \in N} P^{v-1}_{ia} + |N(a, O^{v-1})|(t^v - t^{v-1}) = q_a \right\},$$

$$P^v_{ia} = \begin{cases} P^{v-1}_{ia} + t^v - t^{v-1} & \text{if } i \in N(a, O^{v-1}) \\ P^{v-1}_{ia} & \text{otherwise} \end{cases},$$

where for any set S, $|S|$ denotes the cardinality of S. Since $\hat{O}$ is a finite set, there exists $\bar{v}$ such that $t^{\bar{v}} = 1$. We define $PS(\succ) := P^{\bar{v}}$ as the probabilistic serial assignment for the preference profile $\succ$.

The intuition for the recursive definition above is as follows. Stage $v = 1, \dots$ of the eating algorithm begins at time t^{v-1} with share $\sum_{i \in N} P^{v-1}_{ia}$ of object $a \in O$ having been eaten already. O^{v-1} denotes the set of object types that have not been completely consumed by time t^{v-1}. Each agent in $N(a, O^{v-1})$ eats a, which is his most preferred object in O^{v-1}, until its entire quota q_a is consumed.

Bogomolnaia and Moulin (2001) establish that the probabilistic serial assignment is ordinally efficient and envy-free in their setting (a random assignment is *envy-free* if every agent weakly prefers his own assignment to that of any other agent in terms of first-order stochastic dominance with respect to his reported ordinal preferences). The proofs can be easily adapted to our setting. Neither ordinal efficiency nor envy-freeness is satisfied by the extensively used *random serial dictatorship* mechanism (Abdulkadiroğlu and Sönmez 1998), also known as the *random priority* mechanism (Bogomolnaia and Moulin 2001).

However, the high degree of efficiency and fairness of the probabilistic serial mechanism is not without cost. The mechanism is *not strategy-proof*, that is, an agent is sometimes better off misstating his preferences. In fact, a result of Bogomolnaia and Moulin (2001) implies that there is no mechanism satisfying strategy-proofness, ordinal efficiency and envy-freeness. The ordinal efficiency and envy-freeness of the probabilistic serial mechanism are based on the presumption that agents report their ordinal preferences truthfully. If agents misreport preferences, then neither of the two desirable properties is guaranteed. Therefore, it is important to identify conditions under which agents have incentives to report their ordinal preferences truthfully in the probabilistic serial mechanism.

3. Result

We show that agents have incentives to report ordinal preferences truthfully in the probabilistic serial mechanism when the quota of each object is sufficiently large.

Theorem 1. Let u_i be an expected utility function consistent with a preference $\succ_i$.

(i) There exists M such that if $q_a \geq M$ for all $a \in \hat{O}$, then

$$u_i(PS(\succ_i, \succ_{N\setminus\{i\}})) \geq u_i(PS(\succ'_i, \succ_{N\setminus\{i\}}))$$

for any preference $\succ'_i$, any set of agents $N \ni i$, and any preference profile $\succ_{N\setminus\{i\}}$.

(ii) Claim (i) is satisfied for $M = xD/d$, where $x \approx 1.76322$ solves $x\ln(x) = 1$, $D = \max_{a \succeq_i b \succeq_i \varnothing} u_i(a) - u_i(b)$, and $d = \min_{a \succ_i b, a \succeq_i \varnothing} u_i(a) - u_i(b)$.

A formal proof of the theorem is presented in the Appendix. For a sketch of the argument, fix a preference profile $\succ$, and denote by $\succ' = (\succ'_i, \succ_{N\setminus\{i\}})$ the preference profile where agent i reports $\succ'_i$ instead of $\succ_i$. By deviating from $\succ_i$ to $\succ'_i$, agent i may influence the outcome of the eating algorithm through the following two channels:

- at any instance in the algorithm, *for a fixed set of available objects*, reporting $\succ'_i$ may prevent i from eating his $\succ_i$-most preferred available object
- reporting $\succ'_i$ can *influence the availability schedule of the objects*, e.g., reporting an object as less desirable may lengthen the period when it is available, and further affect the eating behavior of other agents, which in turn can change the times when other objects are available.

The former channel is always detrimental to i, but the latter may be favorable. We prove that i's benefit from the latter channel is smaller than his cost from the former when the quota of each object becomes large.

More specifically, suppose that over some time interval $[t, t')$ agent i eats object a under $\succ'$ and object b under $\succ$, and $a \succ_i b$. It must be that a is not available under $\succ$ at t (otherwise i would be eating a). The proof shows that the share of a available at t under $\succ'$ is small. Since a large part of the q_a share of a is consumed under $\succ'$ before t, if q_a is large, then many agents must eat a over $[t, t')$ under $\succ'$. Hence a cannot be available under $\succ'$ long after t. Therefore, the interval $[t, t')$ must be short. We establish that the size of the interval $[t, t')$ is of an order of magnitude smaller than the sum, denoted by λ, of the lengths of time intervals on which agent i's consumption in the eating algorithm under $\succ$ is $\succ_i$-preferred to that under $\succ'$.

Suppose that $q_a \geq M$ for all $a \in \hat{O}$. In Sections B.2 and B.3 of the Appendix we find lower bounds on M sufficient for truth-telling to be a weakly dominant strategy for agent i. Let $k = |\{a \in \hat{O} | a \succ_i \emptyset\}|$ denote the number of object types that are $\succ_i$-preferred to the null object.

Section B.2 provides a rough bound. Based on the intuition above, we show that the sum of the lengths of time intervals on which i benefits from reporting $\succ'_i$ rather than $\succ_i$ does not exceed $\lambda((1 + 1/M)^k - 1)$. Hence i's expected utility gain from misreporting preferences over these intervals is at most $D\lambda((1 + 1/M)^k - 1)$. At the same time, i's expected utility loss over the intervals where his consumption under $\succ$ is $\succ_i$-preferred to that under $\succ'$ is at least $d\lambda$. Therefore,

$$u_i(PS(\succ)) - u_i(PS(\succ')) \quad \geq \quad d\lambda - D\lambda\left(\left(1+\frac{1}{M}\right)^k - 1\right).$$

The right hand side of the latter inequality is non-negative, and hence truth-telling is a weakly dominant strategy for agent i, if

$$M \geq (k+1)\frac{D}{d}. \tag{1}$$

Section B.3 refines the bound. Let

$$\Lambda = \frac{\lambda}{M}\left(1+\frac{1}{M}\right)^{k-1}.$$

The key observation is that the object i eats at any time t under $\succ$ is $\succeq_i$-preferred to that he eats at $t+\Lambda$ under $\succ'$. Then we can evaluate i's expected utility gain from reporting $\succ_i'$ rather than $\succ_i$ using a translation by Λ of his eating schedule under $\succ$ with respect to that under $\succ'$. We show that i's benefit from misreporting preferences does not exceed the integral of the utility difference between his $\succ_i$-most preferred object and his consumption under $\succ$ over the time interval $[1-\Lambda, 1]$. This leads to a bound on i's expected utility gain from misreporting preferences of $D\Lambda$. Hence,

$$u_i(PS(\succ)) - u_i(PS(\succ')) \geq d\lambda - D\frac{\lambda}{M}\left(1+\frac{1}{M}\right)^{k-1}.$$

A sufficient condition for the right hand side of the inequality above be nonnegative, and truth-telling be a weakly dominant strategy for agent i, is

$$M \geq x\frac{D}{d}. \tag{2}$$

Note that the upper bound on i's expected gain from misreporting his preferences is of order $D\lambda/M$ in Section B.3, but of order $D\lambda k/M$ in Section B.2. Consequently, (2) provides a weaker sufficient condition than (1).

Clearly, $D/d \geq k$, and if the utility difference between no two consecutively ranked objects varies substantially, then D/d is close to k. For instance, consider a school choice setting where student i finds only 10 schools acceptable, and his utility difference between any two consecutively ranked schools is constant. In this case $D/d = 10$. If there are at least 18 seats at every school, then truth-telling is a weakly dominant strategy for i in the probabilistic serial mechanism.

One important feature of the bound (2) is that it is independent of the misstated ordinal preferences $\succ_i'$, the set of agents $N \setminus \{i\}$ and their preference profile $\succ_{N\setminus\{i\}}$. In particular, agent i can verify whether (2) holds using only his information about D/d. Therefore, whenever (2) holds, truth-telling is a best response for i in the probabilistic serial mechanism independently of how many other agents participate and what preferences they report. Even when the quotas are not sufficiently large to make truth-telling a weakly dominant strategy for all agents, truth-telling may be a weakly dominant strategy for some of them.

In the statement of Theorem 1 the condition "$q_a \geq M$ for all $a \in \hat{O}$" can be replaced with "$q_a \geq M$ for all $a \succ_i \emptyset$." Theorem 1 has the following corollary.

Corollary 1. *Suppose that the set $\hat{O}$ of proper object types and the set $\mathcal{U}$ of expected utility functions on lotteries over O are fixed and finite. There exists M such that if $q_a \geq M$ for all $a \in \hat{O}$, then for any set of participating agents, truth-telling is a weakly dominant strategy in the probabilistic serial mechanism for every agent whose utility function is in $\mathcal{U}$.*

Corollary 1 implies that the probabilistic serial mechanism becomes strategy-proof in large allocation problems where the expected utility functions of all agents belong to a given finite set. The latter assertion includes the special case of replica economies. Consider a problem $\Gamma = (N, (\succ_i)_{i\in N}, \hat{O}, (q_a)_{a\in\hat{O}})$ and an expected utility u_i consistent with $\succ_i$ for each i in N. For any positive integer M, the *M-fold replica economy* of $(\Gamma, (u_i)_{i\in N})$ is a random assignment problem in which there are M "replicas" of each agent i with a common utility function u_i, and there are Mq_a copies of each object a in $\hat{O}$. A consequence of the assertion above is that for sufficiently large M, truth-telling is a weakly dominant strategy for every agent in the probabilistic serial mechanism for the M-fold replica of $(\Gamma, (u_i)_{i\in N})$.

4. Example

We present an example that serves three purposes. First, it illustrates some of the ideas of the proof of Theorem 1. Second, it shows that the bound from part (ii) of Theorem 1 cannot be improved by a factor greater than $x \approx 1.76322$. Third, it shows that the conclusion of the theorem cannot be strengthened to claim the existence of M such that if $q_a \geq M$ for all $a \in \hat{O}$, then truth-telling is a weakly dominant strategy for agent i in the probabilistic serial mechanism for every expected utility function u_i. That is, the order of quantifiers $\forall u_i, \exists M$ cannot be replaced with $\exists M, \forall u_i$.

Consider a setting with 2 types of proper objects, a and b, each having quota M. Fix $D > d > 0$ and an agent i with utility index u_i given by $u_i(a) = D, u_i(b) = D - d, u_i(\varnothing) = 0$. Note that u_i is consistent with the ordinal preference $\succ_i$ specified by $a \succ_i b \succ_i \varnothing$. Denote by $\succ'_i$ the preference for agent i with $b \succ'_i a \succ'_i \varnothing$.

Let $N = \{i\} \cup N' \cup N''$ be the set of agents, with N' and N'' of cardinalities M and $M+1$, respectively. Assume that the preferences of the agents in $N' \cup N''$ are as follows:

$$a \succ_j \varnothing \succ_j b, \ \forall j \in N',$$

$$b \succ_j \varnothing \succ_j a, \ \forall j \in N''.$$

Suppose that each agent $j \neq i$ reports $\succ_j$ in the eating algorithm. If i reports $\succ_i$, then he eats object a in the time interval $[0, M/(M+1))$ and the null object in $[M/(M+1), 1]$. If i reports $\succ'_i$ instead of $\succ_i$, then he eats b in $[0, M/(M+2))$, a in $[M/(M+2), M(M+3)/(M+1)(M+2))$, and then $\varnothing$ in $[M(M+3)/(M+1)(M+2), 1]$. Figure 1 depicts the eating schedules for agent i under the preference profiles $\succ = (\succ_i)_{i \in N}$ and $\succ' = (\succ'_i, \succ_{N' \cup N''})$.

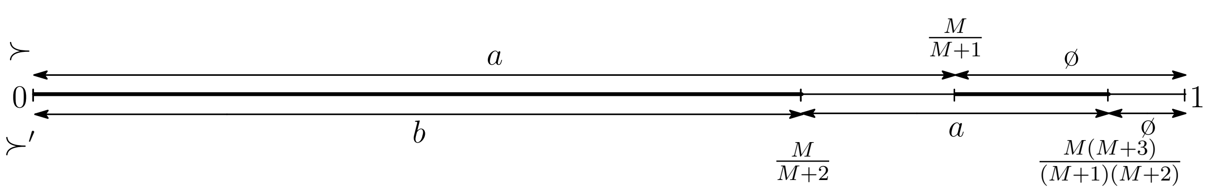

Figure 1. Eating schedules for agent i under $\succ$ and $\succ'$.

In the time interval $[0, M/(M+2))$, agent i eats a under $\succ$ and b under $\succ'$. His expected utility loss from reporting $\succ'_i$ rather than $\succ_i$ over that interval is $dM/(M+2)$. In $[M/(M+1), M(M+3)/(M+1)(M+2))$, i eats $\varnothing$ under $\succ$ and a under $\succ'$. His expected utility gain from reporting $\succ_i'$ rather than $\succ_i$ over that interval is $DM/(M+1)(M+2)$. At any time outside the two intervals, i eats an identical object under $\succ$ and $\succ'$.

As Figure 1 illustrates, by reporting $\succ_i'$ instead of $\succ_i$, agent i suffers losses over the first thick interval and reaps benefits over the second. Note that the length of the second interval is of order M times smaller than that of the first. Hence the difference in i's expected utility between reporting $\succ_i$ and $\succ_i'$ is

$$\frac{dM}{(M+1)(M+2)}\left(M+1-\frac{D}{d}\right).$$

It follows that truth-telling is a weakly dominant strategy for i if $M \geq D/d - 1$ (the assignment under any preference report other than $\succ_i$ and $\succ_i'$ is first-order stochastically dominated with respect to $\succ_i$ by that under either $\succ_i$ or $\succ_i'$). By contrast, i has incentives to report $\succ_i'$ if $M < D/d - 1$. In particular, the bound from part (ii) of Theorem 1 cannot be improved by a factor greater than x. Furthermore, there exists no M such that if $q_a \geq M$ for all $a \in \hat{O}$, then agent i has incentives to report $\succ_i$ for all $D > d > 0$.

There is a delicate part of the proof which is not captured in this example. The initial change in an agent's eating behavior may induce a chain effect on the availability schedule of several objects. Hence, when an agent misstates his preferences, the first interval where he suffers losses can give rise to multiple intervals where he reaps benefits. This issue is addressed by Lemmata 5, 6 and 7 in the Appendix.

5. Conclusion

Truth-telling is a weakly dominant strategy in the probabilistic serial mechanism when there is a large supply of each object type. This result offers support to the use of the mechanism in applications such as university housing and student placement in schools. A remarkable feature of our result is that truth-telling is an exact weakly dominant strategy as opposed to an "almost dominant strategy," which is common in the literature on asymptotic incentive compatibility. Moreover, for a fixed set of object types and an agent with a given expected utility function, our conclusion holds regardless of the number of other participating agents and their ordinal preferences.

The lower bound on the supply of each object type from Theorem 1 cannot be improved by a factor greater than $x \approx 1.76322$. Whether the bound can be improved to any extent is an open question. Nevertheless, our bound may be sufficiently low to make truth-telling a weakly dominant strategy in the probabilistic serial mechanism for practical allocation problems.

Appendix A. Proof of Proposition 1

Proof. Fix the set of proper object types $\hat{O}$ with corresponding quotas $(q_a)_{a\in\hat{O}}$. Consider a random assignment P for the set of agents N. Let P' be a matrix with rows corresponding to the agents in $N \cup N'$, where N' is a set of n' fictitious agents (not in N), such that $P'_{ia} = P_{ia}$ for all $i \in N$ and $a \in O$, $P'_{ja} = (q_a - \sum_{i\in N} P_{ia})/n'$ for all $j \in N'$ and $a \in \hat{O}$, and $P'_{j\o} = 1 - \sum_{a\in\hat{O}} P'_{ja}$ for all $j \in N'$.[7] For sufficiently large n', all entries of the matrix P' are non-negative. Each row of P' sums to 1, and column a of P' sums to q_a for all $a \in \hat{O}$. Since all rows and columns have integer sums and each entry is non-negative, the procedure described by Hylland and Zeckhauser (1979) in the section "Conduct of the Lottery" may be adapted to the current setting to find a convex decomposition of P' into deterministic assignments for the agents in $N \cup N'$. Obviously, the restriction of this convex decomposition to the agents in N induces a convex decomposition of P into deterministic assignments for the agents in N. □

Appendix B. Proof of Theorem 1

B.1. **Notation.** An *eating function* e describes an eating schedule for each agent, $e_i : [0,1] \to O$ for all $i \in N$; $e_i(t)$ represents the object that agent i is eating at time t. We require that e_i be right-continuous with respect to the discrete topology on O (the topology in which all subsets are open), that is,

$$\forall t \in [0,1), \exists \varepsilon > 0 \text{ such that } e_i(t') = e_i(t),\ \forall t' \in [t, t+\varepsilon).$$

[7]McLennan (2002) uses a similar construction.

For an eating function e, let $n_a(t,e)$ be the number of agents eating from object a at time t and $\nu_a(t,e)$ be the share of object a eaten away by time t, i.e.,[8]

$$\begin{aligned} n_a(t,e) &= |\{i \in N | e_i(t) = a\}|, \\ \nu_a(t,e) &= \int_0^t n_a(s,e) ds. \end{aligned}$$

Note that $\nu_a(\cdot, e)$ is continuous.

For every preference profile $\succ$, let $e^{\succ}$ denote the eating function generated by the eating algorithm when agents report $\succ$. Formally, $e_i^{\succ}(t) = a$ for $t \in [t^{v-1}, t^v)$ if $i \in N(a, O^{v-1})$, for (O^v) and (t^v) constructed in the definition of the probabilistic serial mechanism.

Fix a preference profile $\succ$, and denote by $\succ' = (\succ_i', \succ_{N\setminus\{i\}})$ the preference profile where agent i reports $\succ_i'$ instead of $\succ_i$. Let $\bar{e}$ be the eating function such that

$$\bar{e}_i(t) = \begin{cases} e_i^{\succ}(t) & \text{if } e_i^{\succ}(t) = e_i^{\succ'}(t) \\ \emptyset & \text{otherwise} \end{cases},$$

and at each instance, under $\bar{e}_j$ agent $j \neq i$ is eating from his most preferred object at speed 1 among the ones still available (accounting for agent i's specified eating function $\bar{e}_i$). Note that $\bar{e}_j$ may diverge from $e_j^{\succ}$ or $e_j^{\succ'}$ for $j \neq i$ since the available objects at each time may vary across $\bar{e}$, $e^{\succ}$ and $e^{\succ'}$ due to the different eating behavior adopted by i.

Let $\beta(t), \gamma(t)$, and $\delta(t)$ denote the sums of the lengths of time intervals, before time t, on which agent i's consumption in the eating algorithm is $\succ_i$-preferred, $\succ_i$-less preferred, and different, respectively, when the reported preferences change from $\succ$ to $\succ'$. Formally,

$$\begin{aligned} \beta(t) &= \int_0^t \mathbf{1}_{e_i^{\succ'}(s) \succ_i e_i^{\succ}(s)} ds \\ \gamma(t) &= \int_0^t \mathbf{1}_{e_i^{\succ}(s) \succ_i e_i^{\succ'}(s)} ds \\ \delta(t) &= \beta(t) + \gamma(t), \end{aligned}$$

where for any logical proposition p, $\mathbf{1}_p = 1$ if p is true and $\mathbf{1}_p = 0$ if p is false. Set $\lambda = \gamma(1)$.

Define

$$\{a_1, a_2, \ldots, a_{\bar{l}}\} = \{a \in \hat{O} | \exists t \in [0,1), a = e_i^{\succ'}(t) \succ_i e_i^{\succ}(t)\}$$

[8]It can be shown that $n_a(\cdot, e)$ is Riemann integrable.

as the set of objects that are consumed at some time under $e_i^{\succ'}$ and are $\succ_i$-preferred to the consumption at that time under $e_i^{\succ}$. The set is labeled such that $a_1 \succ_i' a_2 \succ_i' \ldots \succ_i' a_{\bar{l}}$. For $l = 1, 2, \ldots, \bar{l}$, let

$$T_l = \inf\{t | a_l = e_i^{\succ'}(t) \succ_i e_i^{\succ}(t)\}$$

be the first instance t when a_l is consumed under $e_i^{\succ'}$ and is $\succ_i$-preferred to the consumption at t under $e_i^{\succ}$. Clearly, $0 < T_1 < T_2 < \ldots < T_{\bar{l}} < 1$.

Let k denote the number of proper object types that are $\succ_i$-preferred to the null object, $k = |\{a \in \hat{O} | a \succ_i \o\}|$. Note that $\bar{l} \leq k$ since $a_l = e_i^{\succ'}(T_l) \succ_i e_i^{\succ}(T_l) \succeq_i \o$ for all l. Set $T_0 = 0, T_{\bar{l}+1} = 1$ as a technical notation convention.

B.2. **Proof of Part (i).** The proof uses Lemmata 1-6 below.

Lemma 1. For all $t \in [0, 1]$ and $a \in \hat{O}$,

$$\begin{aligned} \nu_a(t, e^{\succ}) &\geq \nu_a(t, \overline{e}) \\ \nu_a(t, e^{\succ'}) &\geq \nu_a(t, \overline{e}). \end{aligned}$$

Proof. By symmetry, we only need to prove the first inequality. We proceed by contradiction. Assume that there exist t and a such that $\nu_a(t, e^{\succ}) < \nu_a(t, \overline{e})$. Let

$$t_0 = \inf\{t \in [0, 1] | \exists a \in \hat{O}, \nu_a(t, e^{\succ}) < \nu_a(t, \overline{e})\}. \tag{3}$$

By continuity of $\nu_a(\cdot, e^{\succ}) - \nu_a(\cdot, \overline{e})$, it follows that $t_0 < 1$, and

$$\nu_a(t_0, e^{\succ}) - \nu_a(t_0, \overline{e}) \geq 0, \ \forall a \in \hat{O}. \tag{4}$$

This holds trivially if $t_0 = 0$.

One consequence of (4) is that all objects that are not eaten away by time t_0 under $e^{\succ}$ cannot be eaten away by t_0 under $\overline{e}$ either. Hence the set of objects available at t_0 under $e^{\succ}$ is included in that under $\overline{e}$. It must be that if agent $j \in N$ is eating object $a \in \hat{O}$ at t_0 under $\overline{e}$ and a is available at t_0 under $e^{\succ}$, then j is eating a at t_0 under $e^{\succ}$. Formally,

$$\forall j \in N, \ \overline{e}_j(t_0) = a \neq \o \ \& \ \nu_a(t_0, e^{\succ}) < q_a \Rightarrow e_j^{\succ}(t_0) = a.$$

For $j = i$ the latter step follows from the definition of $\overline{e}$. Therefore,

$$\forall a \in \hat{O},\ \nu_a(t_0, e^{\succ}) < q_a \Rightarrow n_a(t_0, e^{\succ}) \geq n_a(t_0, \overline{e}). \tag{5}$$

Given the right-continuity of $e^{\succ}$ and $\overline{e}$, for sufficiently small $\varepsilon > 0$, we have that for all $t \in [t_0, t_0 + \varepsilon)$ and $a \in \hat{O}$,

$$\begin{aligned} \nu_a(t, e^{\succ}) &= \nu_a(t_0, e^{\succ}) + n_a(t_0, e^{\succ})(t - t_0) \\ \nu_a(t, \overline{e}) &= \nu_a(t_0, \overline{e}) + n_a(t_0, \overline{e})(t - t_0). \end{aligned}$$

Using (4) and (5) we obtain $\nu_a(t, e^{\succ}) \geq \nu_a(t, \overline{e})$ for all $t \in [t_0, t_0 + \varepsilon)$ and $a \in \hat{O}$ with $\nu_a(t_0, e^{\succ}) < q_a$. Note that if $\nu_a(t_0, e^{\succ}) = q_a$ then the inequality $\nu_a(t, e^{\succ}) \geq \nu_a(t, \overline{e})$ holds trivially for all $t \geq t_0$.

By (3), $\nu_a(t, e^{\succ}) \geq \nu_a(t, \overline{e})$ for all $t \in [0, t_0)$ and $a \in \hat{O}$. The arguments above establish that $\nu_a(t, e^{\succ}) \geq \nu_a(t, \overline{e})$ for all $t \in [0, t_0 + \varepsilon)$ and $a \in \hat{O}$, which contradicts the definition of t_0. □

Lemma 2. For all $t \in [0, 1]$,

$$\nu_{\o}(t, e^{\succ}) - \nu_{\o}(t, \overline{e}) \geq -\delta(t).$$

Proof. Note that

$$\nu_{\o}(t, e^{\succ}) - \nu_{\o}(t, \overline{e}) + \delta(t) = \int_0^t [n_{\o}(s, e^{\succ}) - n_{\o}(s, \overline{e}) + \mathbf{1}_{e_i^{\succ}(s) \neq e_i^{\succ\prime}(s)}] ds.$$

Since $\nu_a(t, e^{\succ}) \geq \nu_a(t, \overline{e})$ for all $a \in \hat{O}$ and $t \in [0, 1]$ by Lemma 1, an argument similar to Lemma 1 leads to

$$\begin{aligned} e_i^{\succ}(s) \neq \overline{e}_i(s) &\Rightarrow n_{\o}(s, e^{\succ}) \geq n_{\o}(s, \overline{e}) - 1 \\ e_i^{\succ}(s) = \overline{e}_i(s) &\Rightarrow n_{\o}(s, e^{\succ}) \geq n_{\o}(s, \overline{e}). \end{aligned}$$

Thus the integrand $n_{\o}(s, e^{\succ}) - n_{\o}(s, \overline{e}) + \mathbf{1}_{e_i^{\succ}(s) \neq e_i^{\succ\prime}(s)}$ is non-negative for all $s \in [0, t]$, which completes the proof. □

Lemma 3. For all $t \in [0, 1]$ and $a \in \hat{O}$,

$$\nu_a(t, e^{\succ}) - \nu_a(t, \overline{e}) \leq \delta(t).$$

Proof. The inequality follows immediately from Lemmata 1 and 2, noting that

$$\sum_{a \in O} \nu_a(t, e^{\succ}) - \nu_a(t, \overline{e}) = 0, \forall t \in [0, 1].$$

□

Lemma 4. For all $t \in [0, 1]$ and $a \in \hat{O}$,

$$\nu_a(t, e^{\succ}) - \nu_a(t, e^{\succ'}) \leq \delta(t).$$

Proof. The inequality follows from Lemmata 1 and 3, writing

$$\nu_a(t, e^{\succ}) - \nu_a(t, e^{\succ'}) = [\nu_a(t, e^{\succ}) - \nu_a(t, \overline{e})] - [\nu_a(t, e^{\succ'}) - \nu_a(t, \overline{e})].$$

□

Lemma 5. For all $l = 1, \ldots, \bar{l}$,

$$\beta(T_{l+1}) - \beta(T_l) \leq \frac{\delta(T_l)}{q_{a_l}}.$$

Proof. Since $a_l = e_i^{\succ'}(T_l) \succ_i e_i^{\succ}(T_l)$, it follows that the object a_l is not available at time T_l under the eating function $e^{\succ}$, i.e., $\nu_{a_l}(T_l, e^{\succ}) = q_{a_l}$. By Lemma 4,

$$\nu_{a_l}(T_l, e^{\succ'}) \geq \nu_{a_l}(T_l, e^{\succ}) - \delta(T_l) > q_{a_l} - 1. \tag{6}$$

As $n_{a_l}(\cdot, e^{\succ'})$ is increasing on the time interval where a_l is available under $e^{\succ'}$,

$$n_{a_l}(T_l, e^{\succ'}) > n_{a_l}(T_l, e^{\succ'})T_l \geq \int_0^{T_l} n_{a_l}(s, e^{\succ'})ds = \nu_{a_l}(T_l, e^{\succ'}) > q_{a_l} - 1.$$

Then $n_{a_l}(T_l, e^{\succ'}) \geq q_{a_l}$ because $n_{a_l}(T_l, e^{\succ'})$ is an integer. It follows that $n_{a_l}(s, e^{\succ'}) \geq q_{a_l}$ for all times $s \geq T_l$ when a_l is still available under $e^{\succ'}$. Note that a_l is available under $e^{\succ'}$ at $s \geq T_l$ if $e_i^{\succ'}(s) = a_l$. Therefore,

$$n_{a_l}(s, e^{\succ'}) \geq q_{a_l} \mathbf{1}_{e_i^{\succ'}(s) = a_l}, \forall s \in [T_l, T_{l+1}).$$

By (6), $\nu_{a_l}(T_l, e^{\succ'}) \geq \nu_{a_l}(T_l, e^{\succ}) - \delta(T_l) = q_{a_l} - \delta(T_l)$. Thus

$$\begin{aligned}\delta(T_l) &\geq q_{a_l} - \nu_{a_l}(T_l, e^{\succ'}) \\ &\geq \nu_{a_l}(T_{l+1}, e^{\succ'}) - \nu_{a_l}(T_l, e^{\succ'}) \\ &= \int_{T_l}^{T_{l+1}} n_{a_l}(s, e^{\succ'}) ds \\ &\geq q_{a_l} \int_{T_l}^{T_{l+1}} \mathbf{1}_{e_i^{\succ'}(s)=a_l} ds \\ &= q_{a_l}(\beta(T_{l+1}) - \beta(T_l)),\end{aligned}$$

where the last equality holds because, by the definition of a_l and T_l, the times in $[T_l, T_{l+1})$ when agent i's consumption in the eating algorithm is $\succ_i$-preferred if the reported preferences change from $\succ$ to $\succ'$ are exactly those when i eats a_l. Thus

$$\beta(T_{l+1}) - \beta(T_l) \leq \frac{\delta(T_l)}{q_{a_l}}.$$

□

Lemma 6. If $q_a \geq M$ for all $a \in \hat{O}$, then

$$\beta(T_{l+1}) - \beta(T_l) \leq \frac{\lambda}{M}\left(1 + \frac{1}{M}\right)^{l-1}, \forall l = 0, 1, \ldots, \bar{l}.$$

Proof. We prove the lemma by induction on l. For $l = 0$, the induction hypothesis holds trivially since $\beta(T_1) = 0$.

Let $l \geq 1$. Suppose that the induction hypothesis holds for $0, 1, \ldots, l-1$. We prove that it holds for l.

By the induction hypothesis, if $l \geq 2$,

$$\delta(T_l) \leq \lambda + \sum_{g=1}^{l-1} \beta(T_{g+1}) - \beta(T_g) \leq \lambda + \frac{\lambda}{M} \sum_{g=1}^{l-1} \left(1 + \frac{1}{M}\right)^{g-1} = \lambda \left(1 + \frac{1}{M}\right)^{l-1}. \tag{7}$$

The inequality can be checked separately for $l = 1$.

Since $q_{a_l} \geq M$ by assumption, Lemma 5 and (7) imply that

$$\beta(T_{l+1}) - \beta(T_l) \leq \frac{\lambda}{M}\left(1 + \frac{1}{M}\right)^{l-1},$$

finishing the proof of the induction step. □

Proof of Part (i). Assume that $q_a \geq M$ for all $a \in \hat{O}$. If we set $d = \min_{a \succ_i b, a \succeq_i \o} u_i(a) - u_i(b)$ and $D = \max_{a \succeq_i b \succeq_i \o} u_i(a) - u_i(b)$, then

$$u_i(PS(\succ)) - u_i(PS(\succ')) = \int_0^1 u_i(e_i^{\succ}(s)) - u_i(e_i^{\succ'}(s))ds \geq d\gamma(1) - D\beta(1).$$

By definition, $\gamma(1) = \lambda$. Since $\bar{l} \leq k$ and $\beta(T_1) = 0$, adding up the inequalities from Lemma 6 for $l = 1, 2, \ldots, \bar{l}$, we obtain

$$\beta(1) \leq \sum_{g=0}^{\bar{l}-1} \frac{\lambda}{M}\left(1 + \frac{1}{M}\right)^g = \lambda\left(\left(1 + \frac{1}{M}\right)^{\bar{l}} - 1\right) \leq \lambda\left(\left(1 + \frac{1}{M}\right)^k - 1\right).$$

Therefore,

$$u_i(PS(\succ)) - u_i(PS(\succ')) \;\; \geq \;\; \lambda\left(d - D\left(\left(1 + \frac{1}{M}\right)^k - 1\right)\right),$$

which is non-negative if

$$M \geq \frac{1}{\left(\frac{d}{D} + 1\right)^{1/k} - 1}. \tag{8}$$

This completes the proof. □

B.2.1. *Linearization of the bound (8) as a function of k and D/d.* Using Taylor expansions of $(1+x)^{1/k} - 1$ at $x = 0$, we obtain the inequalities[9]

$$\frac{d}{D}\frac{1}{k} - \frac{1}{2}\left(\frac{d}{D}\right)^2 \frac{1}{k}\left(1 - \frac{1}{k}\right) \leq \left(\frac{d}{D} + 1\right)^{1/k} - 1 \leq \frac{d}{D}\frac{1}{k},$$

which lead to a tight bound for the denominator in (8). Hence truth-telling is a weakly dominant strategy for i if

$$M \geq \frac{D}{d}\frac{k}{1 - \frac{1}{2} \cdot \frac{d}{D}\left(1 - \frac{1}{k}\right)}.$$

As $d/D \leq 1/k$, it follows that

$$k + 1 > \frac{k}{1 - \frac{1}{2} \cdot \frac{d}{D}\left(1 - \frac{1}{k}\right)}.$$

Therefore, truth-telling is a weakly dominant strategy for i if

$$M \geq (k+1)\frac{D}{d}.$$

[9]These inequalities can be verified by taking first and second order derivatives of $(1+x)^{1/k} - 1 - \frac{1}{k}x$ and $(1+x)^{1/k} - 1 - \frac{1}{k}x + \frac{1}{2} \cdot \frac{1}{k}(1 - \frac{1}{k})x^2$ for $x \geq 0$.

B.3. **Proof of Part (ii).** Define

$$\Lambda = \frac{\lambda}{M}\left(1+\frac{1}{M}\right)^{k-1}.$$

Lemma 7. Suppose $q_a \geq M$ for all $a \in \hat{O}$. Then for all $a \in \hat{O}$ and $t \leq T_{\bar{l}}$ with $t+\Lambda \leq 1$,

$$\nu_a(t, e^{\succ}) = q_a \Rightarrow \nu_a(t+\Lambda, e^{\succ'}) = q_a.$$

Proof. Assume that $a \in \hat{O}$ and $t \leq T_{\bar{l}}$ satisfy $t+\Lambda \leq 1$ and $\nu_a(t, e^{\succ}) = q_a$. By inequality (7) in the proof of Lemma 6,

$$\delta(t) \leq \delta(T_{\bar{l}}) \leq M\Lambda. \tag{9}$$

By Lemma 4,

$$\nu_a(t, e^{\succ'}) \geq \nu_a(t, e^{\succ}) - \delta(t) > q_a - 1. \tag{10}$$

Define $t' = t + \Lambda$. We prove that $\nu_a(t', e^{\succ'}) = q_a$ by contradiction. Assume that $\nu_a(t', e^{\succ'}) < q_a$. Note that $n_a(\cdot, e^{\succ'})$ is increasing on the time interval where a is available under $e^{\succ'}$, hence by (10),

$$n_a(t, e^{\succ'}) > n_a(t, e^{\succ'})t \geq \int_0^t n_a(s, e^{\succ'})ds = \nu_a(t, e^{\succ'}) > q_a - 1.$$

It must be that $n_a(t, e^{\succ'}) \geq q_a$ because $n_a(t, e^{\succ'})$ is an integer. Since a is still available at t' under $e^{\succ'}$, it follows that

$$n_a(s, e^{\succ'}) \geq q_a, \forall s \in [t, t').$$

By (9) and (10),

$$\nu_a(t, e^{\succ'}) \geq \nu_a(t, e^{\succ}) - \delta(t) \geq \nu_a(t, e^{\succ}) - M\Lambda = q_a - M\Lambda > \nu_a(t', e^{\succ'}) - M\Lambda.$$

Therefore,

$$M\Lambda > \nu_a(t', e^{\succ'}) - \nu_a(t, e^{\succ'}) = \int_t^{t'} n_a(s, e^{\succ'})ds \geq q_a(t'-t) = q_a\Lambda,$$

which contradicts $q_a \geq M$. $\square$

Proof of Part (ii). Assume that $q_a \geq M$ for all $a \in \hat{O}$. The construction of the sequence (a_l, T_l) and the consequence of Lemma 7 that $\nu_{a_{\bar{l}}}(T_{\bar{l}} + \Lambda, e^{\succ'}) = q_{a_{\bar{l}}}$ if $T_{\bar{l}} + \Lambda \leq 1$ lead to $u_i(e_i^{\succ'}(s)) \leq u_i(e_i^{\succ}(s))$ for all $s > \min\{T_{\bar{l}} + \Lambda, 1\}$.

For technical purposes, we extend the definition of $e_i^{\succ}$ such that $e_i^{\succ}(s) = e_i^{\succ}(0)$ for all $s \in [-\Lambda, 0)$. It follows from Lemma 7 and the observation above that $u_i(e_i^{\succ'}(s)) \leq u_i(e_i^{\succ}(s - \Lambda))$ for all $s \in [0, 1]$. We obtain

$$\begin{aligned}
u_i(PS(\succ)) - u_i(PS(\succ')) &= \int_0^1 u_i(e_i^{\succ}(s)) - u_i(e_i^{\succ'}(s))ds \\
&= \int_0^1 \max\{0, u_i(e_i^{\succ}(s)) - u_i(e_i^{\succ'}(s))\}ds \\
&\quad + \int_0^1 \min\{0, u_i(e_i^{\succ}(s)) - u_i(e_i^{\succ'}(s))\}ds \\
&\geq d\lambda + \int_0^1 \min\{0, u_i(e_i^{\succ}(s)) - u_i(e_i^{\succ}(s - \Lambda))\}ds \\
&= d\lambda + \int_0^1 u_i(e_i^{\succ}(s)) - u_i(e_i^{\succ}(s - \Lambda))ds \\
&= d\lambda + \int_0^1 u_i(e_i^{\succ}(s))ds - \int_{-\Lambda}^{1-\Lambda} u_i(e_i^{\succ}(s))ds \\
&= d\lambda + \int_{1-\Lambda}^1 u_i(e_i^{\succ}(s))ds - \int_{-\Lambda}^{0} u_i(e_i^{\succ}(s))ds \\
&= d\lambda - \int_{-\Lambda}^{0} u_i(e_i^{\succ}(s)) - u_i(e_i^{\succ}(s+1))ds \\
&\geq d\lambda - D\Lambda.
\end{aligned}$$

Therefore,

$$u_i(PS(\succ)) - u_i(PS(\succ')) \geq d\lambda - D\Lambda = \frac{d\lambda}{M}\left(M - \frac{D}{d}\left(1 + \frac{1}{M}\right)^{k-1}\right).$$

Suppose that $M \geq xD/d$, where $x \approx 1.76322$ solves $x\ln(x) = 1$. Let $\mathbf{e} \approx 2.71828$ denote the base of the natural logarithm. Since $D/d \geq k$,

$$\left(1 + \frac{1}{M}\right)^{k-1} < \left(1 + \frac{1}{xk}\right)^{k} = \left(\left(1 + \frac{1}{xk}\right)^{xk}\right)^{1/x} < \mathbf{e}^{1/x}.$$

As $x = \mathbf{e}^{1/x}$, it follows that

$$u_i(PS(\succ)) - u_i(PS(\succ')) \geq \frac{d\lambda}{M}\left(x\frac{D}{d} - \frac{D}{d}\mathbf{e}^{1/x}\right) = 0.$$

Hence truth-telling is a weakly dominant strategy for i if

$$M \geq x\frac{D}{d}.$$

□

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ASYMPTOTIC EQUIVALENCE OF PROBABILISTIC SERIAL AND RANDOM PRIORITY MECHANISMS

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Abstract. The random priority (random serial dictatorship) mechanism is a common method for assigning objects. The mechanism is easy to implement and strategy-proof. However this mechanism is inefficient, for all agents may be made better off by another mechanism that increases their chances of obtaining more preferred objects. This form of inefficiency is eliminated by a mechanism called probabilistic serial, but this mechanism is not strategy-proof. Thus, which mechanism to employ in practical applications is an open question. We show that these mechanisms become equivalent when the market becomes large. More specifically, given a set of object types, the random assignments in these mechanisms converge to each other as the number of copies of each object type approaches infinity. Thus, the inefficiency of the random priority mechanism becomes small in large markets. Our result gives some rationale for the common use of the random priority mechanism in practical problems such as student placement in public schools.
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1. Introduction

Consider a mechanism design problem of assigning indivisible objects to agents who can consume at most one object each. University housing allocation, public housing allocation, office assignment, and student placement in public schools are real-life examples.[1] A typical goal of the mechanism designer is to assign the objects efficiently and fairly. The mechanism often needs to satisfy other constraints as well. For example, monetary transfers may be impossible or undesirable to use, as in the case of low income housing or student placement in public schools. In such a case, random assignments are employed to achieve fairness. Further, the assignment often depends on agents' reports of ordinal preferences over objects rather than full cardinal preferences, as in student placement in public schools in many cities.[2] Two mechanisms are regarded as promising solutions: the

[1]See Abdulkadiroğlu and Sönmez (1999) and Chen and Sönmez (2002) for application to house allocation, and Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003b) for student placement. For the latter application, Abdulkadiroğlu, Pathak, and Roth (2005) and Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) discuss practical considerations in designing student placement mechanisms in New York City and Boston.

[2]Why only ordinal preferences are used in many assignment rules seems unclear, and explaining it is outside the scope of this paper. Following the literature, we take it as given instead. Still, one reason may be that elicitation of cardinal preferences may be difficult (the pseudo-market mechanism proposed by Hylland and Zeckhauser (1979) is one of the few existing mechanisms incorporating cardinal preferences over objects.) Another reason may be that efficiency based on ordinal preferences is well justified regardless of agents' preferences; many theories of preferences over random outcomes (not just

random priority (RP) mechanism and the probabilistic serial (PS) mechanism (Bogomolnaia and Moulin 2001).[3]

In random priority, agents are ordered with equal probability and, for each realization of the ordering, the first agent in the ordering receives her favorite (the most preferred) object, the next agent receives his favorite object among the remaining ones, and so on. Random priority is strategy-proof, that is, reporting ordinal preferences truthfully is a weakly dominant strategy for every agent. Moreover, random priority is ex-post efficient, that is, the lottery over deterministic assignments produced by it puts positive probability only on Pareto efficient deterministic assignments.[4] The random priority mechanism can also be easily tailored to accommodate other features, such as students applying as roommates in college housing,[5] or respecting priorities of existing tenants in house allocation (Abdulkadiroğlu and Sönmez 1999) and non-strict priorities by schools in student placement (Abdulkadiroğlu, Pathak, and Roth 2005, Abdulkadiroğlu, Pathak, Roth, and Sönmez 2005).

Perhaps more importantly for practical purposes, the random priority mechanism is straightforward and transparent, with the lottery used for assignment specified explicitly. Transparency of a mechanism can be crucial for ensuring fairness in the eyes of participants, who may otherwise be concerned about possible "covert selection."[6] These

expected utility theory) agree that people prefer one assignment over another if the former first-order stochastically dominates the latter.

[3]Priority mechanisms are studied for divisible object allocation by Satterthwaite and Sonnenschein (1981) and then indivisible object allocation by Svensson (1994). Abdulkadiroğlu and Sönmez (1998) study the random priority mechanism as an explicitly random assignment mechanism.

[4]Abdulkadiroğlu and Sönmez (2003a) point out that random assignment that is induced by an ex post efficient lottery may also be induced by an ex post inefficient lottery. On the other hand, random priority as implemented in common practice produces an ex post efficient lottery since, for any realization of agent ordering, the assignment is Pareto efficient.

[5]Applications by would-be roommates can be easily incorporated into the random priority mechanism by requiring each group to receive the same random priority order. For instance, non-freshman undergraduate students at Columbia University can apply as a group, in which case they draw the same lottery number. The lottery number, along with their seniority points, determines their priority. If no suite is available to accommodate the group or they do not like the available suite options, they can split up and make choices as individuals. This sort of flexibility between group and individual assignments seems difficult to achieve in other mechanisms such as the probabilistic serial mechanism.

[6]The concern of covert selection was pronounced in UK schools, which led to adoption of a new Mandatory Admission Code in 2007. The code, among other things, "makes the admissions system more straightforward, transparent and easier to understand for parents" ("Schools admissions code to end covert selection," *Education Guardian*, January 9, 2007). There had been numerous appeals by

advantages explain the wide use of the random priority mechanism in many settings, such as house allocation in universities and student placement in public schools.

Despite its many advantages, the random priority mechanism may entail unambiguous efficiency loss ex ante. Adapting an example by Bogomolnaia and Moulin (2001), suppose that there are two types of objects a and b with one copy each and the "null object" $\o$ representing the outside option. There are four agents $1, 2, 3$ and 4, where agents 1 and 2 prefer a to b to $\o$ while agents 3 and 4 prefer b to a to $\o$. One can compute the assignment for each of $4! = 24$ possible agent orderings, and the resulting random assignments are given by Table 1.[7] From the table it can be seen that each agent ends up with her less preferred object with positive probability in this economy. This is because two agents of the same preference type may get the first two positions in the ordering, in which case the second agent will take her non-favorite object.[8] Obviously, any two agents of different preferences can benefit from trading off the probability share of the non-favorite object with that of the favorite. In other words, the random priority assignment has unambiguous efficiency loss. For instance, every agent prefers an alternative random assignment in Table 2.

	Object a	**Object** b	**Object** $\o$
Agents 1 and 2	5/12	1/12	1/2
Agents 3 and 4	1/12	5/12	1/2

Table 1. Random assignments under RP.

	Object a	**Object** b	**Object** $\o$
Agents 1 and 2	1/2	0	1/2
Agents 3 and 4	0	1/2	1/2

Table 2. Random assignments preferred to RP by all agents.

A random assignment is said to be ordinally efficient if it is not first-order stochastically dominated for all agents by any other random assignment. Ordinal efficiency is

parents on schools assignments in the UK; there were 78,670 appeals in 2005-2006, and 56,610 appeals in 2006-2007.

[7]Each entry of the table specifies the allocation probability for an agent-object pair. For example, the number $\frac{5}{12}$ in the upper left entry means that each of agents 1 and 2 receives object a with probability $\frac{5}{12}$.

[8]For instance, if agents are ordered by $1, 2, 3$ and 4, then 1 gets a, 2 gets b, and 3 and 4 get $\o$.

perhaps the most relevant efficiency concept in the context of assignment mechanisms based solely on ordinal preferences. The example implies that random priority may result in an ordinally inefficient random assignment.

The probabilistic serial mechanism introduced by Bogomolnaia and Moulin (2001) eliminates the inefficiency present in RP. Imagine that each indivisible object is a divisible object of probability shares: If an agent receives fraction p of an object, we interpret that she receives the object with probability p. Given reported preferences, consider the following "eating algorithm." Time runs continuously from 0 to 1. At every point in time, each agent "eats" her favorite object with speed one among those that have not been completely eaten away. At time $t = 1$, each agent is endowed with probability shares of objects. The PS assignment is defined as the resulting probability shares. In the current example, agents 1 and 2 start eating a and agents 3 and 4 start eating b at $t = 0$ in the eating algorithm. Since two agents are consuming one unit of each object, both a and b are eaten away at time $t = \frac{1}{2}$. As no (proper) object remains, agents consume the null object between $t = \frac{1}{2}$ and $t = 1$. Thus the resulting PS assignment is given by Table 2. In particular, the probabilistic serial mechanism eliminates the inefficiency that was present under RP. More generally, the probabilistic serial random assignment is ordinally efficient if all the agents report their ordinal preferences truthfully.

The probabilistic serial mechanism is not strategy-proof, however. In other words, an agent may receive a more desirable random assignment (with respect to her true expected utility function) by misreporting her ordinal preferences. The mechanism is also less straightforward and less transparent for the participants than random priority, since the lottery used for implementing the random assignment can be complicated and is not explicitly specified. The tradeoffs between the two mechanisms — random priority and probabilistic serial — are not easy to evaluate, hence the choice between the two remains an important outstanding question in practical applications. Indeed, Bogomolnaia and Moulin (2001) show that no mechanism satisfies ordinal efficiency, strategy-proofness, and symmetry (equal treatment of equals) in all finite economies with at least four objects and agents. Thus one cannot hope to resolve the tradeoffs by finding a mechanism with these three desiderata. Naturally, the previous studies have focused only on the choice between random priority and probabilistic serial.

The contribution of this paper is to offer a new perspective on the tradeoffs between the random priority and probabilistic serial mechanisms. We do so by showing that the two mechanisms become virtually equivalent in large markets. Specifically, we demonstrate that, given a set of arbitrary object types, the random assignments in these mechanisms converge to each other, as the number of copies of each object type approaches infinity.

To see our result in a concrete example, consider replicas of the above economy where, in the q-fold replica economy, there are q copies of a and b and there are $2q$ agents who prefer a to b to $\emptyset$ and $2q$ who prefer b to a to $\emptyset$. Clearly, agents receive the same random assignment in PS for all replica economies. By contrast, the market size makes a difference in RP. Figure 1 plots the misallocation probability in RP, i.e., the probability that an agent of each type receives the non-favorite proper object, as a function of the market size q.[9] The misallocation probability accounts for the only difference in random assignment between RP and PS in this example. As can be seen from the figure, the misallocation probability is positive for all q but declines and approaches zero as q becomes large.

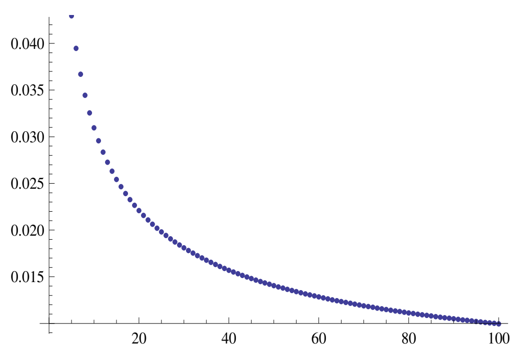

FIGURE 1. Relationship between the market size and the random assignment in RP. The horizontal axis measures market size q while the vertical axis measures the misallocation probability.

Hence the difference between RP and PS becomes small in this specific example. The main contribution of this paper is to demonstrate the asymptotic equivalence more generally (beyond the simple cases of replica economies) and understand its economics.

Our result has several implications. First, it implies that the inefficiency of the random priority mechanism becomes small and disappears in the limit, as the economy becomes large. Second, the result implies that the incentive problem of the probabilistic serial mechanism disappears in large economies. Taken together, these implications mean that we do not have as strong a theoretical basis for distinguishing the two mechanisms in large markets as in small markets; indeed, both will be good candidates in large markets

[9]The misallocation probability is, for example, the probability that agents who prefer a to b receive b.

since they have good incentive, efficiency, and fairness properties.[10] Given its practical merit, though, our result lends some support for the common use of the random priority mechanism in practical applications, such as student placement in public schools.

In our model, the large market assumption means that there exist a large number of copies of each object type. This model includes several interesting cases. For instance, a special case is the replica economies model wherein the copies of object types and of agent types are replicated repeatedly. Considering large economies as formalized in this paper is useful for many practical applications. In student placement in public schools, there are typically a large number of identical seats at each school. In the context of university housing allocation, the set of rooms may be partitioned into a number of categories by building and size, and all rooms of the same type may be treated to be identical.[11] Our model may be applicable to these markets.

Our equivalence result is obtained in the limit of finite economies. As it turns out, this result is tight in the sense that we cannot generally expect the two mechanisms to be equivalent in any finite economies (Proposition 3 in Section 6). What it implies is that their difference becomes aribitrary small as the economy becomes sufficiently large.

We obtain several further results. First, we present a model with a continuum of agents and continuum of copies of (finite) object types. We show that the random priority and probabilistic serial assignments in finite economies converge to the corresponding assignments in the continuum economy. In that sense, the limit behavior of these mechanisms in finite economies is captured by the continuum economy. This result provides a foundation for modeling approaches that study economies with a continuum of objects and agents directly.

Second, we consider a situation in which individual participants are uncertain about the population distribution of preferences, so they do not necessarily know the popularity of each object *even in the large market*. It turns out that the random priority and probabilistic serial mechanisms are asymptotically equivalent even in the presence of such aggregate uncertainty, but the resulting assignments are not generally ordinally efficient

[10] As mentioned above, Bogomolnaia and Moulin (2001) present three desirable properties, namely ordinal efficiency, strategy-proofness, and equal treatment of equals, and show that no mechanism satisfies all these three desiderata in finite economies. Random priority satisfies all but ordinal efficiency while probabilistic serial satisfies all but strategy-proofness. Our equivalence result implies that both mechanisms satisfy all these desiderata in the limit economy, thus overcoming impossibility in general finite economies.

[11] For example, the assignment of graduate housing at Harvard University is based on the preferences of each student over eight types of rooms: two possible sizes (large and small) and four buildings.

even in the large market. This inefficiency is not unique to these mechanisms, however. We show a general impossibility result that there exists no (symmetric) mechanism that is strategy-proof and ordinally efficient (even) in the continuum economy.

Finally, we show that both mechanisms can be usefully applied to, and their large-market equivalence holds in, cases where different groups of agents are treated differently, where different types of objects have different numbers of copies, and where agents demand multiple objects.

The rest of the paper proceeds as follows. Section 2 discusses related literature. Section 3 introduces the model. Section 4 defines the random priority mechanism and the probabilistic serial mechanism. Sections 5 and 6 present the main results. Section 7 investigates further topics. Section 8 concludes. Proofs are found in the Appendix unless stated otherwise.

2. Related literature

Pathak (2006) compares random priority and probabilistic serial using data on the assignment of about 8,000 students in the public school system of New York City. He finds that many students obtain a better random assignment in the probabilistic serial mechanism but that the difference is small. The current paper complements his study by explaining why the two mechanisms are not expected to differ much in some school choice settings.

Kojima and Manea (2008) find that reporting true preferences becomes a dominant strategy for each agent under probabilistic serial when there are a large number of copies of each object type. Their paper and ours complement each other both substantively and methodologically. Substantively, Kojima and Manea (2008) suggest that probabilistic serial may be more useful than random priority in applications but do not analyze how random priority behaves in large economies. The current paper addresses that question and provides a clear large-market comparison of the two mechanisms, showing that the main deficiency of random priority, inefficiency, is reduced in large economies. Furthermore, our analysis provides intuition for their result.[12] To see this point, first recall that truthtelling is a dominant strategy in random priority. Since our result shows that probabilistic serial is close to random priority in a large economy, this observation suggests that it is difficult to profitably manipulate the probabilistic serial mechanism. Methodologically, we note that our asymptotic equivalence is based on the assumption that agents

[12]However, the result of Kojima and Manea (2008) cannot be derived from the current paper since they establish a dominant strategy result in a large but finite economies, while our equivalence result holds only in the limit as the market size approaches infinity.

report preferences truthfully both in random priority and probabilistic serial. The result of Kojima and Manea (2008) gives justification to this assumption by showing that truthtelling is a dominant strategy under probabilistic serial in large finite economies.

Manea (2006) considers environments in which preferences are randomly generated and shows that the probability that the random priority assignment is ordinally inefficient approaches one as the market becomes large under a number of assumptions. He obtains the results in two environments one of which is comparable to ours and one of which differs from ours in that the number of object types grows to infinity as the economy becomes large. In either case, his result does not contradict ours because of a number of differences. Most importantly, Manea (2006) focuses on whether there is *any* ordinal inefficiency in the random priority assignment, while the current paper investigates *how much* difference there is between the random priority and the probabilistic serial mechanisms, and hence (indirectly) how much ordinal inefficiency the random priority mechanism entails. As we show in Proposition 3, this distinction is important particularly for the welfare assessment of RP.

While the analysis of large markets is relatively new in matching and resource allocation problems, it has a long tradition in many areas of economics. For example, Roberts and Postlewaite (1976) show that, under some conditions, the Walrasian mechanism is difficult to manipulate in large exchange economies.[13] Similarly, incentive properties of a large class of double auction mechanisms are studied by, among others, Gresik and Satterthwaite (1989), Rustichini, Satterthwaite, and Williams (1994), and Cripps and Swinkels (2006). Two-sided matching is an area closely related to our model. In that context, Roth and Peranson (1999), Immorlica and Mahdian (2005), and Kojima and Pathak (2008) show that the deferred acceptance algorithm proposed by Gale and Shapley (1962) becomes increasingly hard to manipulate as the number of participants becomes large. Many of these papers show particular properties of given mechanisms, such as incentive compatibility and efficiency. One of the notable features of the current paper is that we show the equivalence of apparently dissimilar mechanisms, beyond specific properties of each mechanism.

Finally, our paper is part of a growing literature on random assignment mechanisms.[14] The probabilistic serial mechanism is generalized to allow for weak preferences, existing property rights, and multi-unit demand by Katta and Sethuraman (2006), Yilmaz (2006),

[13] See also Jackson (1992) and Jackson and Manelli (1997).

[14] Characterizations of ordinal efficiency are given by Abdulkadiroğlu and Sönmez (2003a) and McLennan (2002).

and Kojima (2008), respectively. Kesten (2008) introduces two mechanisms, one of which is motivated by the random priority mechanism, and shows that these mechanisms are equivalent to the probabilistic serial mechanism. In the scheduling problem (a special case of the current environment), Crès and Moulin (2001) show that the probabilistic serial mechanism is group strategy-proof and ordinally dominates the random priority mechanism but these two mechanisms converge to each other as the market size approaches infinity, and Bogomolnaia and Moulin (2002) give two characterizations of the probabilistic serial mechanism.

3. Model

For each $q \in \mathbb{N}$, consider a **q-economy**, $\Gamma^q = (N^q, (\pi_i)_{i \in N^q}, O)$, where N^q represents the set of agents and O represents the set of **proper object types** (we assume that O is identical for all q). There are $|O| = n \in \mathbb{N}$ object types, and each object type $a \in O$ has **quota** q, that is, q copies of a are available.[15] There exist an infinite number of copies of a **null object** $\emptyset$, which is not included in O. Let $\tilde{O} := O \cup \{\emptyset\}$. Each agent $i \in N^q$ has a **strict preference** $\pi_i \in \Pi$ over $\tilde{O}$. More specifically, $\pi_i(a) \in \{1, \dots, n+1\}$ is the ranking of a according to agent i's preference $\pi_i \in \Pi$, that is, agent i prefers a to b if and only if $\pi_i(a) < \pi_i(b)$. For any $O' \subset \tilde{O}$,

$$Ch_\pi(O') := \{a \in O' | \pi(a) \leq \pi(b) \ \forall b \in O'\},$$

is the favorite object among O' for type π-agents (agents whose preference type is π).

The set N^q of agents is partitioned into different preference types $\{N^q_\pi\}_{\pi \in \Pi}$, where N^q_π is the set of the agents with preference $\pi \in \Pi$ in the q-economy. Let $m^q_\pi := \frac{|N^q_\pi|}{q}$ be the per-unit number of agents of type π in the q-economy. We assume, for each $\pi \in \Pi$, there exists $m^\infty_\pi \in \mathbb{R}_+$ such that $m^q_\pi \to m^\infty_\pi$ as $q \to \infty$. For $q \in \mathbb{N} \cup \{\infty\}$, let $m^q := \{m^q_\pi\}_{\pi \in \Pi}$. Throughout, we do not impose any restriction on the way in which the q-economy, Γ^q, grows with q (except for the existence of the limit $m^\infty_\pi = \lim_{q \to \infty} m^q_\pi$ for each $\pi \in \Pi$).

A special case of interest is when the economy grows at a constant rate with q. We say that the family $\{\Gamma^q\}_{q \in \mathbb{N}}$ are **replica economies** if $m^q_\pi = m^\infty_\pi$ (or equivalently, $|N^q_\pi| = q|N^1_\pi|$) for all $q \in \mathbb{N}$ and all $\pi \in \Pi$, and call Γ^1 a **base economy** and Γ^q its **q-fold replica**.

Fix any $q \in \mathbb{N}$. Throughout the paper, we focus on random assignments that are **symmetric** in the sense that the agents with the same preference type π receive the same lottery over the objects.[16] Formally, a **random assignment in the q-economy** is a mapping $\phi^q : \Pi \to \Delta\tilde{O}$, where $\Delta\tilde{O}$ is the set of probability distributions over $\tilde{O}$, that

[15] Given a set X, we denote the cardinality of X by $|X|$ or $\#X$.

[16] This property is often called the "equal treatment of equals" axiom.

satisfies the feasibility constraint $\sum_{\pi \in \Pi} \phi^q_a(\pi) \cdot |N^q_\pi| \leq q$, for each $a \in O$, where $\phi^q_a(\pi)$ represents the probability that a type π-agent receives the object a.

3.1. **Ordinal Efficiency.** Consider a q-economy where $q \in \mathbb{N}$. A random assignment ϕ^q **ordinally dominates** another random assignment $\hat{\phi}^q$ **at** m^q if, for each preference type π with $m^q_\pi > 0$, the lottery $\phi^q(\pi)$ first-order stochastically dominates the lottery $\hat{\phi}^q(\pi)$,

$$\sum_{\pi(b) \leq \pi(a)} \phi^q_b(\pi) \geq \sum_{\pi(b) \leq \pi(a)} \hat{\phi}^q_b(\pi) \qquad \forall \pi, m^q_\pi > 0, \forall a \in \tilde{O}, \tag{3.1}$$

with strict inequality for some (π, a). Random assignment ϕ^q is **ordinally efficient at** m^q if it is not ordinally dominated at m^q by any other random assignment.[17] If ϕ^q ordinally dominates $\hat{\phi}^q$ at m^q, then every agent of every preference type prefers her assignment under ϕ^q to the one under $\hat{\phi}^q$ according to any expected utility function with utility index consistent with their ordinal preferences.

We say that ϕ^q is **individually rational at** m^q if there exists no preference type $\pi \in \Pi$ with $m^q_\pi > 0$ and object $a \in O$ such that $\phi^q_a(\pi) > 0$ and $\pi(\emptyset) < \pi(a)$. That is, individual rationality requires that no agent be assigned an unacceptable object with positive probability. A random assignment is ordinally inefficient unless it is individually rational, since an agent receiving unacceptable objects can be assigned the null object instead without hurting any other agent.

We say that ϕ^q is **non-wasteful at** m^q if there exists no preference type $\pi \in \Pi$ with $m^q_\pi > 0$ and objects $a \in O, b \in \tilde{O}$ such that $\pi(a) < \pi(b)$, $\phi^q_b(\pi) > 0$ and $\sum_{\pi' \in \Pi} \phi^q_a(\pi') m^q_{\pi'} < 1$. That is, non-wastefulness requires that there be no object which some agent prefers to what she consumes but is not fully consumed. If there were such an object, the allocation would be ordinally inefficient.

Consider the binary relation $\triangleright(\phi^q, m^q)$ on O defined by

$$a \triangleright (\phi^q, m^q)\, b \iff \exists \pi \in \Pi, m^q_\pi > 0, \pi(a) < \pi(b) \text{ and } \phi^q_b(\pi) > 0. \tag{3.2}$$

That is, $a \triangleright (\phi^q, m^q)\, b$ if there are some agents who prefer a to b but are assigned to b with positive probability. If a relation $\triangleright(\phi^q, m^q)$ admits a cycle, then the relevant agents can

[17]As noted before, this paper focuses on symmetric random assignments. We note that an ordinally efficient random assignment is not ordinally dominated by any possibly asymmetric random assignment (this property is defined as ordinal efficiency by Bogomolnaia and Moulin (2001)). To show this claim by contraposition, assume a symmetric random assignment ϕ is ordinally dominated by some asymmetric random assignment ϕ'. Define another random assignment ϕ'' by giving each agent the average of assignments for agents of the same type as hers in ϕ'. Assignment ϕ'' is symmetric by definition and ordinally dominates ϕ since ϕ' does, showing the claim.

trade off shares of non-favorite objects along the cycle and all do better, so the allocation would be ordinally inefficient.

One can show that ordinal efficiency is equivalent to acyclicity of this binary relation, individual rationality, and non-wastefulness. This is shown by Bogomolnaia and Moulin in a setting in which each object has quota 1, there exist an equal number of agents and objects, and all objects are acceptable to all agents.[18] Their characterization extends straightforwardly to our setting as follows (so the proof is omitted).

Proposition 1. The random assignment ϕ^q is ordinally efficient at m^q if and only if the relation $\rhd(\phi^q, m^q)$ is acyclic and ϕ^q is individually rational and non-wasteful at m^q.

4. Two Competing Mechanisms: Random Priority and Probabilistic Serial

4.1. **Probabilistic Serial Mechanism.** We first describe the **probabilistic serial** mechanism, which is an adaptation of the mechanism proposed by Bogomolnaia and Moulin to our setting. The idea is to regard each object as a divisible object of "probability shares." Each agent "eats" a probability share of the best available object with speed one at every time $t \in [0, 1]$ (object a is available at time t if not all q shares of a have been eaten by time t).[19] The resulting profile of object shares eaten by agents by time 1 obviously induces a random assignment, which we call the **probabilistic serial random assignment**.

To formally describe the assignment under the probabilistic serial mechanism, for any $q \in \mathbb{N} \cup \{\infty\}$, $O' \subset \tilde{O}$ and $a \in O' \setminus \{\o\}$, let

$$m_a^q(O') := \sum_{\pi \in \Pi : a \in Ch_\pi(O')} m_\pi^q,$$

be the per-unit number of agents whose favorite (most preferred) object in O' is a in the q-economy, and let $m_{\o}^q(O') := 0$ for all $q \in \mathbb{N} \cup \{\infty\}$ and $O' \subset \tilde{O}$. Now fix a q-economy Γ^q. The PS assignment is then defined by the following sequence of steps. For step $v = 0$, let $O^q(0) = \tilde{O}$, $t^q(0) = 0$, and $x_a^q(0) = 0$ for every $a \in \tilde{O}$. Given $O^q(0), t^q(0)$, $\{x_a^q(0)\}_{a \in \tilde{O}}, \ldots,$

[18]This restriction implies that individual rationality and non-wastefulness are trivially satisfied by every feasible random assignment.

[19]Bogomolnaia and Moulin (2001) consider a broader class of simultaneous eating algorithms, where eating speeds may vary across agents and time.

$O^q(v-1), t^q(v-1), \{x_a^q(v-1)\}_{a\in\tilde{O}}$, for each $a \in \tilde{O}$, define for step v

$$t_a^q(v) = \sup\left\{t \in [0,1] \,|\, x_a^q(v-1) + m_a^q(O^q(v-1))(t - t^q(v-1)) < 1\right\}, \tag{4.1}$$

$$t^q(v) = \min_{a\in O(v-1)} t_a^q(v), \tag{4.2}$$

$$O^q(v) = O^q(v-1) \setminus \{a \in O^q(v-1) | t_a^q(v) = t^q(v)\}, \tag{4.3}$$

$$x_a^q(v) = x_a^q(v-1) + m_a^q(O^q(v-1))(t^q(v) - t^q(v-1)), \tag{4.4}$$

with the terminal step defined as $\bar{v}^q := \min\{v' | t^q(v') = 1\}$.

These recursive equations are explained as follows. Step $v = 1, ...$ begins at time $t^q(v-1)$ with the share $x_a^q(v-1)$ of object $a \in O$ having been eaten already, and a set $O^q(v-1)$ of object types remaining to be eaten. Object $a \in O^q(v-1)$ will be the favorite among the remaining objects to $q \cdot m_a^q(O^q(v-1))$ agents, so they will start eating a until its entire remaining quota $q(1 - x_a^q(v-1))$ is gone. The eating of a will go on, unless step v ends, until time $t_a^q(v)$ at which point the entire share of object a is consumed away or time runs out (see (4.1)). Step v ends at $t^q(v)$ when the first of the remaining objects disappears or time runs out (see (4.2)). Step $v+1$ begins at that time, with the remaining set $O^q(v)$ of objects adjusted for the expiration of some object(s) (see (4.3)) and the remaining share $x_a^q(v)$ adjusted to reflect the amount of a consumed during step v (see (4.4)). This process is complete when time $t = 1$ is reached, and involves at most $|\tilde{O}|$ steps.

For each $a \in \tilde{O}$, we define its **expiration date** $T_a^q := \{t^q(v) | t^q(v) = t_a^q(v) \text{ for some } v\}$ to be the time at which the eating of a is complete.[20] Note that the expiration dates are all deterministic. The expiration dates completely pin down the random assignment for the agents. Let $\tau_a^q(\pi) := \min\{T_a^q, \max\{T_b^q | \pi(b) < \pi(a), b \in O\}\}$ be the expiration date of the last object that a type π-agent prefers to a (if it is smaller than T_a^q, and T_a^q otherwise). Each type-π agent starts eating a at time $\tau_a^q(\pi)$ and consumes the object until it expires at time T_a^q. Hence, a type π-agent's probability of getting assigned to $a \in \tilde{O}$ is simply its duration of consumption; i.e., $PS_a^q(\pi) = T_a^q - \tau_a^q(\pi)$.

Following Bogomolnaia and Moulin (2001), we can show that PS^q is ordinally efficient. First, individual rationality follows since no agent ever consumes an object less preferred than the null object. Next, non-wastefulness follows since, if an object say a is not completely consumed then $T_a^q = 1$, so no agent type will ever consume any object she prefers less than a. Finally, if an agent type prefers a to b but consumes b with positive probability, then it must be that $T_a^q < T_b^q$, or else she will never consume b. This means

[20]Expiration date T_a^q for each $a \in \tilde{O}$ is well defined. If a good a runs out for some step $v < \bar{v}^q$, then $T_a^q = t^q(v) = t_a^q(v)$. If a good a never runs out, then $T_a^q = t^q(\bar{v}^q) = t_a^q(\bar{v}^q) = 1$.

that $\rhd(PS^q, m^q)$ is acyclic since the expiration dates are linearly ordered. That the expiration dates are deterministic (so their orders are not random) is therefore a key feature that makes PS ordinally efficient.

Proposition 2. For any $q \in \mathbb{N}$, PS^q is ordinally efficient.

One main drawback of the probabilistic serial mechanism, as identified by Bogomolnaia and Moulin (2001), is that it is not strategy-proof. In other words, an agent may be better off by reporting a false ordinal preference.

4.2. **Random Priority Mechanism.** In the **random priority** mechanism (Bogomolnaia and Moulin 2001) (known also as the **random serial dictatorship** by Abdulkadiroğlu and Sönmez (1998)), the agents are randomly ordered, and each agent successively claims (or more precisely is assigned to) her favorite object among the remaining ones, following that order. Our key methodological innovation is to develop a "temporal" reinterpretation of RP so as to facilitate its comparison with PS. Imagine first each agent i draws a lottery number f_i from $[0,1]$ independently and uniformly. Imagine next that time runs from 0 to 1 just as in PS, and agent i "arrives" *at time* f_i and claims her favorite object among those *available at that time.* It is straightforward to see that this alternative definition is equivalent to the original one. (The agents are assigned sequentially almost always since no two lottery draws coincide with positive probability).

Let RP^q denote the random assignment resulting from the random priority mechanism in Γ^q. Our temporal reinterpretation of RP allows us to formulate RP^q via recursive equations much like (4.1)-(4.4). To begin, fix any agent i (of any type π) and ask whether any particular object a is available to her given any possible lottery number she may draw. This can be answered by studying how long that object would last in our time frame $[0,1]$ *if agent* i *were absent.* This can be done by characterizing the "expiration date" of each object in the "hypothetical" economy with $|N^q|-1$ agents with preferences $\pi_{-i} \in \Pi^{(|N^q|-1)}$ and lottery numbers $f_{-i} = (f_j)_{j\in N\setminus\{i\}} \in [0,1]^{(|N^q|-1)}$. It will be later explained how studying this economy allows us to compute i's random assignment in the (real) q-economy.

First, define $\hat{m}^q_{\pi'}(t,t') := \frac{\#\{j\in N^q_{\pi'}\setminus\{i\} | f_j \in (t,t']\}}{q}$ to be the per-unit number of agents of type π' (except i if $\pi' = \pi$) whose lottery draws lie in $(t,t']$. For any $O' \subset \tilde{O}$ and $a \in O' \setminus \{\o\}$, let

$$\hat{m}^q_a(O';t,t') := \sum_{\pi'\in\Pi: a\in Ch_{\pi'}(O')} \hat{m}^q_{\pi'}(t,t'),$$

be the per-unit number of agents in $N^q \setminus \{i\}$ whose favorite object in O' is a and whose lottery draws are in $(t,t']$. Let $m^q_{\o}(O';t,t') := 0$ for all $q \in \mathbb{N}\cup\{\infty\}$ and $O' \subset \tilde{O}$.

Then, the expiration dates of the objects in this hypothetical economy are described as follows, given (π_{-i}, f_{-i}). Let $\hat{O}^q(0) = \tilde{O}$, $\hat{t}^q(0) = 0$, and $\hat{x}^q_a(0) = 0$ for every $a \in \tilde{O}$. Given $\hat{O}^q(0), \hat{t}^q(0), \{\hat{x}^q_a(0)\}_{a\in\tilde{O}}, \ldots, \hat{O}^q(v-1), \hat{t}^q(v-1), \{\hat{x}^q_a(v-1)\}_{a\in\tilde{O}}$, for each $a \in \tilde{O}$, define

$$\hat{t}^q_a(v) = \sup\left\{t \in [0,1] \,\middle|\, \hat{x}^q_a(v-1) + \hat{m}^q_a(\hat{O}^q(v-1); \hat{t}^q(v-1), t) < 1\right\}, \tag{4.5}$$

$$\hat{t}^q(v) = \min_{a\in\hat{O}(v-1)} \hat{t}^q_a(v), \tag{4.6}$$

$$\hat{O}^q(v) = \hat{O}^q(v-1) \setminus \{a \in \hat{O}^q(v-1) | \hat{t}^q_a(v) = \hat{t}^q(v)\}, \tag{4.7}$$

$$\hat{x}^q_a(v) = \hat{x}^q_a(v-1) + \hat{m}^q_a(\hat{O}^q(v-1); \hat{t}^q(v-1), \hat{t}^q(v)), \tag{4.8}$$

with the terminal step defined as $\tilde{v}^q := \min\{v' | \hat{t}^q(v') = 1\}$.

These equations are explained in much the same way as (4.1)-(4.4). Step $v = 1, \ldots$ begins at time $\hat{t}^q(v-1)$ with the share $\hat{x}^q_a(v-1)$ of object $a \in O$ having been claimed already, and a set $\hat{O}^q(v-1)$ of objects remaining to be claimed. There are $q \cdot \hat{m}^q_a(\hat{O}^q(v-1); \hat{t}^q(v-1), t)$ agents whose favorite object is a, and who arrive during the time span $[\hat{t}^q_a(v-1), t]$, so object a lasts until $\hat{t}^q_a(v)$ defined by (4.5), unless step v ends beforehand. Step v ends at $\hat{t}^q(v)$ when the first of the remaining objects disappears or time runs out, as defined by (4.6). Step $v+1$ begins at that time, with the remaining set $\hat{O}^q(v)$ of object types adjusted for the expiration of an object (see (4.7)) and the remaining share $\hat{x}^q_a(v)$ adjusted to reflect the amount of the object consumed during step v (see (4.8)). This process is complete when time $t = 1$ is reached, and involves at most $|\tilde{O}|$ steps.

Now re-enter agent i with type π, and consider any object $a \in \tilde{O}$. The object a is available to her if and only if she "arrives" before a **cutoff time** $\hat{T}^q_a := \{\hat{t}^q(v) | \hat{t}^q(v) = \hat{t}^q_a(v), \text{ for some } v\}$, at which the last copy of a would be claimed. At the same time, she will wish to claim a if and only if it becomes her favorite — namely, she arrives after the last object she prefers to a runs out. In sum, a type π-agent obtains a if and only if her lottery draw f_i lands in an interval $[\hat{\tau}^q_a(\pi), \hat{T}^q_a]$, where $\hat{\tau}^q_a(\pi) := \min\{\hat{T}^q_a, \max\{\hat{T}^q_b | \pi(b) < \pi(a), b \in O\}\}$, an event depicted in Figure 2, in case $\hat{\tau}^q_a(\pi) = \hat{T}^q_b$ for some $b \neq a$.

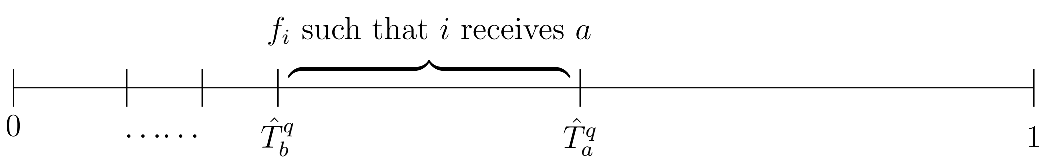

Figure 2: Cutoffs of objects under RP.

Note the cutoff time $\hat{T}^q_a$ of each object a is a random variable since the arrival times f_{-i} of the other agents are random. Therefore, the **random priority random assignment** is defined, for $i \in N^q_\pi$ and $a \in \tilde{O}$, as $RP^q_a(\pi) := \mathbb{E}[\hat{T}^q_a - \hat{\tau}^q_a(\pi)]$, where the expectation $\mathbb{E}$ is taken with respect to $f_{-i} = (f_j)_{j \neq i}$ which are distributed i.i.d uniformly on $[0, 1]$.

The random priority mechanism is widely used in practice, as mentioned in the Introduction. Moreover, the mechanism is **strategy-proof**, that is, reporting true ordinal preferences is a dominant strategy for each agent. Furthermore, it is **ex post efficient**, that is, the assignment after random draws are realized is Pareto efficient. As illustrated in Introduction, however, the mechanism may entail ordinal inefficiency. Ordinal inefficiency of RP can be traced to the fact that the cutoff times of the objects are random and personalized. In the example of Introduction, an agent who prefers a to b may face $\hat{T}^1_a < \hat{T}^1_b$ and the agent who prefers b to a may face $\hat{T}^1_a > \hat{T}^1_b$. In these cases, the agents receive their non-favorite objects with positive probability. Hence both $a \rhd (RP^1, m^1) b$ and $b \rhd (RP^1, m^1) a$ occur, resulting in cyclicity of the relation $\rhd(RP^1, m^1)$. As will be seen, as $q \to \infty$, the cutoff times of the random priority mechanism converge in probability to deterministic limits that are common to all agents, and this feature ensures acyclicity of the binary relation $\rhd$ in the limit.

5. Equivalence of Two Mechanisms in the Continuum Economy

Our ultimate goal is to show that RP^q and PS^q converge to each other as $q \to \infty$. Toward this goal, we first introduce **a continuum economy** in which there exists a unit mass of each object in O and mass m^∞_π of agent type π for each $\pi \in \Pi$. One should think of this continuum economy as a heuristic representation of a large economy which possesses the same demographic profiles (i.e., the limit measures $\{m^\infty_\pi\}_{\pi \in \Pi}$) as the limit of our finite economies but otherwise bears no direct relationship with them. The relevance of this model will be seen in the next section where we show it captures the limit behavior of the finite economies. Specifically, we shall show that the random assignment of the PS and RP defined in this continuum economy coincides with the random assignments arising from these mechanisms in the limit of the q-economies as $q \to \infty$. In this sense, the continuum economy serves as an instrument of our analysis. As will be clear, however, it also brings out the main intuition behind our equivalence result and its implications.

One issue in analyzing a continuum economy is to describe aggregate consequences of randomness at the individual level for a continuum of agents. This issue arises with our RP model given the use of individual lottery drawings, but possibly with other mechanisms as well. Laws of large numbers — a natural tool for dealing with such an issue — can be problematic in this environment.[21] However, a weak law of large numbers developed by

[21]See Judd (1985) for a classic reference for the associated conceptual problems.

Uhlig (1996) turns out to be sufficient for our purpose.[22] Alternatively, one can simply view our constructs as mathematical definitions that conform to plausible large market heuristics.

A random assignment in the continuum economy is defined as a mapping $\phi^* = (\phi^*_a)_{a\in O} : \Pi \to \Delta\tilde{O}$ such that $\sum_{\pi\in\Pi} \phi^*_a(\pi)\cdot m^\infty_\pi \leq 1$ for each $a \in O$. As before, $\phi^*_a(\pi)$ is interpreted as the probability that each (atomless) agent of type π receives object a, and feasibility requires that the total mass of each object consumed not exceed its total quota (unit mass). We now consider the two mechanisms in this economy.

5.1. **Probabilistic serial mechanism.** The PS can be defined in this economy with little modification. The (masses of) agents "eat" probability shares of the objects simultaneously at speed one over time interval $[0,1]$ in the order of their stated preferences. The random assignments are then determined by the duration of eating each object by a given type of agent. As with the finite economy, the **random assignment** PS^* **of probabilistic serial in the continuum economy** is determined by the expiration dates of the objects, i.e., the times at which the objects are all consumed.

Naturally, these expiration dates are defined recursively much as in the PS of finite economies. Let $O^*(0) = \tilde{O}$, $t^*(0) = 0$, and $x^*_a(0) = 0$ for every $a \in \tilde{O}$. Given $O^*(0), t^*(0)$, $\{x^*_a(0)\}_{a\in\tilde{O}}, \ldots, O^*(v-1), t^*(v-1), \{x^*_a(v-1)\}_{a\in\tilde{O}}$, for each $a \in \tilde{O}$, define

$$t^*_a(v) = \sup\left\{t\in[0,1]\,|x^*_a(v-1) + m^\infty_a(O^*(v-1))(t - t^*(v-1)) < 1\right\}, \tag{5.1}$$

$$t^*(v) = \min_{a\in O^*(v-1)} t^*_a(v), \tag{5.2}$$

$$O^*(v) = O^*(v-1)\setminus\{a\in O^*(v-1)|t^*_a(v) = t^*(v)\}, \tag{5.3}$$

$$x^*_a(v) = x^*_a(v-1) + m^\infty_a(O^*(v-1))(t^*(v) - t^*(v-1)), \tag{5.4}$$

with the terminal step defined as $\bar{v}^* := \min\{v'|t^*(v') = 1\}$.

These equations are precisely the same as the corresponding ones (4.1) - (4.4) for the PS of the finite economies, except for the fact that $m^\infty_a(\cdot)$'s replace $m^q_a(\cdot)$'s. The explanations following (4.1) - (4.4) apply here verbatim. The expiration date of each object a defined by $T^*_a = \{t^*_a(v)|t^*_a(v) = t^*(v) \text{ for some } v\}$ determines the random assignment PS^* of probabilistic serial in the continuum economy in the same manner as in finite economies.

[22]This version of law of large numbers ensures that, for a function X mapping $i \in [a,b]$ into an L_2 probability space of random variable with a common mean μ and finite variance σ^2, Riemann integral $\int_a^b X(i)di = \mu$ with probability one (see Theorem 2 of Uhlig (1996)). For convenience, we shall suppress the qualifier "with probability one" in our discussion here.

5.2. **Random priority mechanism.** Defining the random priority mechanism in the continuum economy requires some care. One issue is describing the aggregate behavior of the individual drawings of lotteries, as required in our version of RP. Recall in our RP, each agent draws a lottery number f from $[0, 1]$ according to the uniform distribution. The aggregate distribution of the agents in terms of their lottery numbers then matches the uniform distribution, according to the weak law of large numbers; namely, the measure of agents with lottery numbers f or less among mass m agents will be precisely mf with probability one.[23] The second issue is to define the procedure itself. The finite RP procedure of successively executing individual choice according to lottery numbers cannot work in the continuum economy. We thus define the continuum economy RP as follows:

- Step $v = 1$: For each object $a \in O$, determine a value $\hat{t}^*_a(1) \in [0, 1]$ such that the measure of agents whose favorite object is a and whose lottery numbers are less than $\hat{t}^*_a(1)$ equals one; if no such value exists, let $\hat{t}^*_a(1) = 1$. Assign the agents with lottery numbers less than $\hat{t}^*(1) := \min_a \hat{t}^*_a(1)$ to their favorite objects. If the entire masses of agents are assigned, stop. Or else, remove the assigned objects along with the agents who received them, and iterate to Step $v = 2$.

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

- Step $v = 2, ...$: For each object $a \in O$, determine a value $\hat{t}^*_a(v) \in [0, 1]$ such that the measure of agents whose favorite object among the remaining ones is a and whose lottery numbers are less than $\hat{t}^*_a(v)$ equals the measure of the remaining quota of that object; if no such value exists, let $\hat{t}^*_a(v) = 1$. Assign the agents with lottery numbers less than $\hat{t}^*(v) := \min_a \hat{t}^*_a(v)$ their favorite remaining objects. If the entire masses of agents are assigned, stop. Or else, remove the assigned objects along with the agents who received them, and iterate to Step $v + 1$.

Since there are finite object types, this procedure ends in finite steps. As noted in the previous section, the cutoff time $\hat{T}^*_a$ of each object a, defined by $\hat{T}^*_a = \{\hat{t}^*_a(v) | \hat{t}^*_a(v) = \hat{t}^*(v) \text{ for some } v\}$, determines the random assignment RP^*. Clearly, the above procedure entails recursive equations much like those defined for PS. These equations determine $\hat{t}^*_{[\cdot]}(v), \hat{t}^*(v), \hat{O}^*(v), \hat{x}^*(v)$ — in place of $t^*_{[\cdot]}(v), t^*(v), O^*(v), x^*(v)$ in each step just as before.

Most importantly, they are precisely the same as (5.1)-(5.4), if we let $\hat{O}^*(0) = \tilde{O}$, $\hat{t}^*(0) = 0$, and $\hat{x}^*_a(0) = 0$ for every $a \in \tilde{O}$. This can be shown inductively. Suppose that

[23] Letting $F_U(k) = k$ denote the cdf of the uniform distribution, the weak law of large numbers in Theorem 2 of (Uhlig (1996)) implies that $\int_0^1 1_{\{f \leq h\}} df = F_U(h) = h$ with probability one. Rather than appealing to a law of large numbers, one could instead imbed lottery f as agent's "hidden" type as in Abdulkadiroğlu, Che, and Yasuda (2008).

$\hat{O}^*(v-1) = O^*(v-1)$, $\hat{t}^*(v-1) = t^*(v-1)$, and $\hat{x}^*_a(v-1) = x^*_a(v-1)$, $\forall a \in \tilde{O}$. Consider step v now. With mass $x^*_a(v-1)$ of each object a already claimed, a will be claimed by those agents whose favorite object among $O^*(v-1)$ is a and whose lottery numbers are less than $\hat{t}^*_a(v)$. There is a mass $m^\infty_a(O^*(v-1))[\hat{t}^*_a(v) - t^*(v-1)]$ of such agents. Hence, (5.1) determines $\hat{t}^*_a(v)$ at step v. This means $\hat{t}^*_a(v) = t^*_a(v)$ for all $a \in O$, which in turn implies (5.2), so $\hat{t}^*(v) = t^*(v)$. At the end of step v, then object a such that $t^*_a(v) = t^*(v)$ is completely claimed, so (5.3) holds and a new set $\hat{O}^*(v) = O^*(v)$ of objects remains. Mass $m^\infty_a(O^*(v-1))(t^*(v) - t^*(v-1))$ of each object a is claimed at step v, so the cumulative measure of a claimed by that step will be given by (5.4), implying $\hat{x}^*_a(v) = x^*_a(v)$. The equivalence of the recursive equations of the two mechanisms implies that $\hat{T}^*_a = T^*_a$; namely, the cutoff time of each object under RP matches precisely the expiration date of the same object under PS. As noted above, this means that $RP^* = PS^*$; that is, the random assignments of the two mechanisms are the same.

The intuition for the equivalence can be obtained by invoking our temporal interpretation of RP wherein time runs continuously from 0 to 1 and each agent must claim an object at the time equal to her lottery draw f. From the individual agent's perspective, the mechanisms are still not comparable; an agent consumes a given object for an interval of time in PS, whereas the same agent picks his object outright at a given point of time in RP. Yet, the mechanisms can be compared easily when one looks from the perspective of each object. Each object is consumed over a period of time up to a certain point in both cases. That point is called the expiration date under PS and the cutoff time under RP. Our equivalence argument boils down to the observation that the supply of each object disappears at precisely the same point of time under the two mechanisms. This happens because, for any given interval, the rate at which an object is consumed is the same under both mechanisms. To be concrete, fix an object $a \in O$ and consider the span of time from t to $t + \delta$, for some $\delta > 0$. Suppose the consumption rate of all objects have been the same up to time t under both mechanisms. Say a is the favorite among the remaining objects for mass m of agents. Then, under PS, these agents will eat at speed 1 during that time span, so the total consumption of that object during that time span will be $m \cdot \delta$. Under RP, the same mass m will favor the object among the remaining objects (given the assumption of the same past consumption rates). During that time span, only those with lottery number $f \in [t, t+\delta)$ can arrive to consume. By the weak law of large numbers, a fraction δ of any positive mass arrive during this time span to claim their objects. Hence, mass $m \cdot \delta$ of agents will consume object a during the time span. Our main argument for the proof in the next section is much more complex, yet the same insight will be seen to drive the equivalence result.

Before turning to the main analysis, we point out a few relatively obvious implications of the equivalence obtained for the continuum economy.

- It is straightforward to show that the strategy-proofness of RP extends to this continuum economy. The equivalence established above then means that an agent's assignment probabilities from RP are the same as those from PS, for any ordinal preferences he may report, holding fixed all others' reports. It follows that PS is strategy-proof in the continuum economy.[24]
- It is also straightforward to show the ordinal efficiency of PS in this economy. The equivalence then implies that RP is ordinally efficient.
- The above two observations mean that the impossibility theorem by Bogomolnaia and Moulin (2001) does not extend to the continuum economy: There exists a symmetric mechanism (RP or equivalently PS) that is strategy-proof and ordinally efficient.

6. Asymptotic Equivalence of Two Mechanisms

While the last section demonstrates that RP and PS produce the same random assignment in the continuum economy, it is not clear whether the assignments in large but finite economies are approximated well by the continuum economy. This section will establish that RP and PS assignments in finite economies in fact converge to that in the continuum economy. Not only will this establish asymptotic equivalence of the two mechanisms, but the result will provide a limit justification for the continuum economy studied above.

We first show that PS^q converges to PS^* as $q \to \infty$. The convergence occurs in all standard metrics; for concreteness, we define the metric by $||\phi - \hat{\phi}|| := \sup_{\pi \in \Pi, a \in O} |\phi_a(\pi) - \hat{\phi}_a(\pi)|$ for any pair of random assignments ϕ and $\hat{\phi}$. The convergence of PS^q to PS^* is immediate if $\{\Gamma^q\}_{q \in \mathbb{N}}$ are replica economies. In this case, $m_a^q(O') = m_a^\infty(O')$ for all q and a, so the recursive definitions, (4.1), (4.2), (4.3), and (4.4), of the PS procedure for each q-economy all coincide with those of the continuum economy, namely (5.1), (5.2), (5.3), and (5.4). The other cases are established as well.

Theorem 1. $||PS^q - PS^*|| \to 0$ *as* $q \to \infty$. *Further,* $PS^q = PS^*$ *for all* $q \in \mathbb{N}$ *if* $\{\Gamma^q\}_{q \in \mathbb{N}}$ *are replica economies.*

[24] Here, by strategy-proofness we mean that the random assignment under truthtelling is equal to or first order stochastically dominates the one under false preferences. This property is even stronger than the property shown for PS in large finite economies by Kojima and Manea (2008).

This theorem assumes implicitly that agents report their true preferences under PS in large but finite economies. This assumption can be justified based on Kojima and Manea (2008). Their result implies that, given any finite set of possible cardinal utility types of agents, truthtelling is a dominant strategy under probabilistic serial for any q-economy with sufficiently large (but finite) q. Although we chose not to specify the cardinal utilities of agents in our model for simplicity, their result is directly applicable.[25]

We next show that RP^q converges to $RP^* = PS^*$ as $q \to \infty$.

Theorem 2. $||RP^q - RP^*|| \to 0$ *as* $q \to \infty$.

These theorems show that the random assignment of the two mechanisms in the continuum economy capture their limiting behavior in a large but finite economy. In this sense, they provides a limit justification for an approach that models the mechanisms directly in the continuum economy. More importantly, the asymptotic equivalence follows immediately from these two theorems upon noting that $PS^* = RP^*$.

Corollary 1. $||RP^q - PS^q|| \to 0$ as $q \to \infty$.

The intuition behind the asymptotic equivalence (Corollary 1) is that the expiration dates of the objects under PS and the cutoff times of the corresponding objects under RP converge to each other as the economy grows large. As we argued in the previous section, this follows from the fact that the rates at which the objects are consumed under both mechanisms become identical in the limit. To see this again, fix any time $t \in [t^*(v), t^*(v+1))$ for some v, and fix any object $a \in O$. Under RP^*, assuming that objects $O^*(v)$ are available at time t, the fraction of a consumed during time interval $[t, t+\delta]$ for small δ is $\delta \cdot m_a^\infty(O^*(v))$, namely the measure of those whose favoribe object among $O^*(v)$ is a times the duration of their consumption of a.

In RP^q, assuming again that the same set $O^*(v)$ of objects is available at t, the measure $m_a^q(O^*(v); t, t+\delta)$ of agents (whose favorite among $O^*(v)$ is a) arrive during the (same) time interval $[t, t+\delta]$ and will consume a, so the fraction of a consumed during that interval is $m_a^q(O^*(v); t, t+\delta)$. As $q \to \infty$, this fraction converges to $\delta \cdot m_a^\infty(O^*(v))$, since by a law of large numbers, the arrival rate of these agents approaches $m_a^\infty(O^*(v))$.

The main challenge of the proof is to make this intuition precise when there are intertemporal linkages in the consumption of objects — namely, a change in consumption at

[25] If cardinal utilities of agents are drawn from an infinite types, then for any q some agents may have incentives to misreport preferences. However, even in such a setting the result of Kojima and Manea (2008) implies that the fraction of agents for whom truthtelling is not a dominant strategy converges to zero as $q \to \infty$. Thus the truthtelling assumption in Theorem 1 is justified in this case as well.

one point of time alters the set of available objects, and thus the consumption rates of all objects, at later time. Our proof employs an inductive method to handle these linkages.

Is our asymptotic equivalence tight? In other words, can we generally expect the random assignments of the two mechanisms to coincide in a finite economy? Figure 1 appears to suggest otherwise, showing that the RP and PS assignments remain different for all finite values of q. In fact, this observation can be made quite general in the following sense.

Proposition 3. Consider a family $\{\Gamma^q\}_{q\in\mathbb{N}}$ of replica economies. Then, RP^q is ordinally efficient for some $q \in \mathbb{N}$ if and only if $RP^{q'}$ is ordinally efficient for every $q' \in \mathbb{N}$. That is, for any given base economy, the random priority assignment is ordinally efficient for all replica economies or ordinally inefficient for all of them.

In particular, Proposition 3 implies that the ordinal inefficiency of RP does not disappear completely in any finitely replicated economy if the random priority assignment is ordinally inefficient in the base economy. More importantly, it may be misleading to simply examine whether a mechanism suffers ordinal inefficiencies; even if a mechanism is ordinally inefficient, the magnitude of the inefficiency may be very small, as is the case with RP in large economies.

7. Extensions

7.1. **Group-specific Priorities.** In some applications, the social planner may need to give higher priorities to some agents over others. For example, when allocating graduate dormitory rooms, the housing office at Harvard University assigns rooms to first year students first, and then assigns remaining rooms to existing students. Other schools prioritize housing assignments based on students' seniority and/or their academic performances.[26]

To model such a situation, assume that each student belongs to one of the classes C and, for each class $c \in C$, consider any density function g_c over $[0,1]$. The asymmetric random priority mechanism associated with $g = (g_c)_{c\in C}$ lets each agent i in class c to draw f_i according to the density function g_c independently from others, and the agent with the smallest draw among all agents receives her favorite object, the agent with the second-smallest draw receives his favorite object from the remaining ones, and so forth. The random priority mechanism is a special case in which g_c is a uniform distribution on $[0,1]$ for each $c \in C$. The asymmetric probabilistic serial mechanism associated with g

[26]For instance, Columbia University gives advantage in lottery draw based on seniority in its undergraduate housing assignment. The Technion gives assignment priorities to students based on both seniority and academic performance (Perach, Polak, and Rothblum (2007)). Claremont McKenna College and Pitzer College give students assignment priority based on the number of credits they have earned.

is defined by simply letting agents in class c eat with speed $g_c(t)$ at each time $t \in [0,1]$. The probabilistic serial mechanism is a special case in which g_c is a uniform distribution on $[0,1]$ for each $c \in C$.

For each $q \in \mathbb{N}$, $\pi \in \Pi$ and $c \in C$, let $m^q_{\pi,c}$ be per-unit number of agents in class c of preference type π in the q-economy. If $m^\infty_{\pi,c} := \lim_{q\to\infty} m^q_{\pi,c}$ exists for all π and c, then the asymptotic equivalence generalizes to a general profile of distributions g. In particular, given any g, the asymmetric random priority mechanism associated with g and the asymmetric probabilistic serial mechanism associated with g converge to the same limit as $q \to \infty$. In Appendix D, we provide formal definitions for asymmetric RP and PS in the continuum economy and show their equivalence.

7.2. **Aggregate Uncertainty.** The environment of our model is deterministic in the sense that the supply of objects and preferences of agents are fixed. By contrast, uncertainty in preferences is a prevalent feature in real-life applications. In the context of student placement, for instance, popularity of schools may vary, and students and their parents may know their own preferences but not those of others. Aggregate uncertainty can be incorporated into our model.[27] It turns out that the asymptotic equivalence of RP and PS continues to hold even with aggregate uncertainty. We also point out that a new issue of efficiency arises in this model.

Define Ω to be a finite state space. For any $q \in \mathbb{N}$ and $\omega \in \Omega$, let $\rho^q(\omega)$ be the probability of state ω and $m^q_\pi(\omega)$ be the per-unit number of the agents of preference type π in state ω. Assume (in the same spirit as in the basic model) that there exist well-defined limits $\rho^\infty(\omega) := \lim_{q\to\infty} \rho^q(\omega)$ for all $\omega \in \Omega$ and $m^\infty_\pi(\omega) := \lim_{q\to\infty} m^q_\pi(\omega)$ for all $\pi \in \Pi$ and for all $\omega \in \Omega$. Then, the asymptotic equivalence of RP and PS holds state by state by Corollary 1. Therefore the ex ante random assignments in RP and PS converge to each other as well. Note that this last conclusion follows because Ω is finite, and the ex ante random assignment is simply a weighted average of random assignments across different states. We also note that an exact equivalence holds in the continuum economy for a more general (possibly infinite) state space since the equivalence holds at each state (see Section 5).

Aggregate uncertainty introduces a new issue of efficiency, however, as seen below.

Example 1. Let $\phi^q_a(\pi,\omega)$ be the probability that an agent with preference type π obtains a under state ω in random assignment ϕ^q in the q-economy. Let $O = \{a,b\}$, $\Omega = \{\omega_a, \omega_b\}$, $\rho^q(\omega_a) = \rho^q(\omega_b) = \frac{1}{2}$, agents with preference π^{ab} prefer a to b to $\emptyset$ and those with π^{ba}

[27] We are grateful to an anonymous referee for inspiring us to study the issues presented in this section.

prefer b to a to $\emptyset$. There is measure 4 of agents; 60% of them are of type π^{ab} at state ω_a and 60% of them are of type π^{ba} at state ω_b. More formally, $m^q_{\pi^{ab}}(\omega_a) = \frac{12}{5}, m^q_{\pi^{ba}}(\omega_a) = \frac{8}{5}, m^q_{\pi^{ab}}(\omega_b) = \frac{8}{5}, m^q_{\pi^{ba}}(\omega_b) = \frac{12}{5}$.[28] For each state ω and each agent, the probability that she is of type π is $P(\pi|\omega) := \frac{m^q_\pi(\omega)}{m^q_{\pi^{ab}}(\omega)+m^q_{\pi^{ba}}(\omega)}$. Random assignments under probabilistic serial PS^q can be computed to be

$$PS^q(\pi^{ab},\omega_a) = \left(\frac{5}{12},\frac{1}{12},\frac{1}{2}\right), \qquad PS^q(\pi^{ba},\omega_a) = \left(0,\frac{1}{2},\frac{1}{2}\right),$$
$$PS^q(\pi^{ab},\omega_b) = \left(\frac{1}{2},0,\frac{1}{2}\right), \qquad PS^q(\pi^{ba},\omega_b) = \left(\frac{1}{12},\frac{5}{12},\frac{1}{2}\right).$$

Now consider an agent who knows her preference is π^{ab} (but not the state). From this interim perspective, she forms her posterior belief about the state according to Bayes' law. Specifically, a type π^{ab} agent believes that the state is $\omega = \omega_a, \omega_b$ with probability

$$\bar{P}(\omega|\pi^{ab}) := \frac{\rho^q(\omega)P(\pi^{ab}|\omega)}{\rho^q(\omega_a)P(\pi^{ab}|\omega_a)+\rho^q(\omega_b)P(\pi^{ab}|\omega_b)}.$$

Hence, she expects to receive object a with probability

$$\bar{P}(\omega_a|\pi^{ab})PS^q_a(\pi^{ab},\omega_a) + \bar{P}(\omega_b|\pi^{ab})\cdot PS^q_a(\pi^{ab},\omega_b) = \frac{9}{20}.$$

Similarly, she obtains b with probability $\frac{1}{20}$. By symmetry, a type-π^{ba} agent obtains b and a with probabilities $\frac{9}{20}$ and $\frac{1}{20}$ respectively in PS^q.

Consider now a random assignment ϕ^q,

$$\phi^q(\pi^{ab},\omega_a) = \left(\frac{5}{12},0,\frac{7}{12}\right), \qquad \phi^q(\pi^{ba},\omega_a) = \left(0,\frac{5}{8},\frac{3}{8}\right),$$
$$\phi^q(\pi^{ab},\omega_b) = \left(\frac{5}{8},0,\frac{3}{8}\right), \qquad \phi^q(\pi^{ba},\omega_b) = \left(0,\frac{5}{12},\frac{7}{12}\right),$$

whose feasibility can be shown by calculation. Under ϕ^q, each type of agent receives her favorite object with probability $\frac{1}{2}$ and the null object with probability $\frac{1}{2}$ (i.e., a type-π^{ab} agent obtains a with probability $\frac{1}{2}$, and a type-π^{ba} agent obtains b with probability $\frac{1}{2}$). Therefore, for every agent, her lottery at ϕ^q first-order stochastically dominates the one at PS^q, i.e., ϕ^q ordinally dominates PS^q. Notice the inefficiency does not vanish even as the market size approaches infinity ($q \to \infty$); PS^q does not depend on q in this example. Since RP and PS are asymptotically equivalent, RP remains ordinally inefficient even as $q \to \infty$ as well.

[28] The current example can be seen as generalizing the one discussed in the Introduction. In that example, there is a unique state of the world in which 50 percent of agents are of type π^{ab} and the remaining 50 percent of agents are of type π^{ba}.

One may conclude from this example that, when there is aggregate uncertainty, RP and PS are deficient and an alternative mechanism should replace them. However, there is a sense in which some inefficiencies are not limited to these specific mechanisms but rather inherent in the environment. More specifically, no mechanism is both ordinally efficient and strategy-proof, even in the continuum economy.

To analyze this issue, we formally introduce some concepts. A **mechanism** is a mapping from an environment to a random assignment. To avoid notational clutter, we simply associate a mechanism with the random assignment ϕ^* it induces for a given environment (although the dependence on the environment will be suppressed). Let $\phi^*_a(\pi,\omega)$ be the probability that a type-π agent receives object a at state ω in the continuum economy. Given $\phi^*, a \in \tilde{O}, \pi, \pi' \in \Pi$, let

$$\Phi^*_a(\pi'|\pi) := \frac{\sum_{\omega\in\Omega} \rho^\infty(\omega) P(\pi|\omega) \phi^*_a(\pi',\omega)}{\sum_{\omega\in\Omega} \rho^\infty(\omega) P(\pi|\omega)}$$

be the conditional probability that a type-π agent receives a from mechanism ϕ^* when she reports type π' instead. Let $\Phi^*_a(\pi) := \Phi^*_a(\pi|\pi)$ be the conditional probability that a type-π agent receives a when telling the truth. A mechanism ϕ^* is ordinally efficient if, for any m^∞, there is no random assignment $\hat{\phi}^*$ such that, for each preference type π with $m^\infty_\pi(\omega) > 0$ for some $\omega \in \Omega$, the lottery $(\hat{\Phi}^*_a(\pi))_{a\in\tilde{O}}$ first-order stochastically dominates $(\Phi^*_a(\pi))_{a\in\tilde{O}}$ at m^∞ with respect to π. Mechanism ϕ^* is strategy-proof if, for any m^∞ and any $\pi, \pi' \in \Pi$, $(\Phi^*_a(\pi))_{a\in\tilde{O}}$ at m^∞ is equal to or first-order stochastically dominates $(\Phi^*_a(\pi'|\pi))_{a\in\tilde{O}}$ at m^∞ with respect to preference π.[29]

Proposition 4. In the continuum economy with aggregate uncertainty, there exists no mechanism that is strategy-proof and ordinally efficient.[30]

Note that the statement focuses on the continuum economy. This is without loss of generality since, in finite economies, the impossibility result holds even without aggregate uncertainty (Bogomolnaia and Moulin 2001). Note also that aggregate uncertainty is essential for Proposition 4, since RP (or equivalently PS) satisfies strategy-proofness and ordinal efficiency in the continuum economy if there is no aggregate uncertainty (see Section 5).

[29]The notion of strategy-proofness here is ordinal, just as in Bogomolnaia and Moulin (2001). Note, however, that if a mechanism fails to be strategy-proof in the ordinal sense, it fails to be strategy-proof for some profile of cardinal values.

[30]Note that we presuppose symmetry throughout the paper in the sense that agents with the same preferences receive the same lottery. Without symmetry, a deterministic priority mechanism with a fixed agent ordering across states is both strategy-proof and ordinally efficient.

7.3. **Unequal Number of Copies.** We focused on a setting in which there are q copies of each object type in the q-economy. It is straightforward to extend our results to settings in which there are an unequal number of copies, as long as quotas of object types grow proportionately. More specifically, if there exist positive integers $(q_a)_{a \in O}$ such that the quota of object type a is $q_a q$ in the q-economy, then our results extend with little modification of the proof.

On the other hand, we need *some* assumption about the growth rate of quotas, as the following example shows.

Example 2. Consider an economy Γ^q with 4 types of proper objects, a, b, c, and d, where quotas of a and b stay at one while those of c and d are q. Let $N^q = N^q_{\pi^{ab}} \cup N^q_{\pi^{ba}} \cup N^q_{\pi^{cd}} \cup N^q_{\pi^{dc}}$ be the set of agents, with $|N^q_{\pi^{ab}}| = |N^q_{\pi^{ba}}| = 2, |N^q_{\pi^{cd}}| = |N^q_{\pi^{dc}}| = 2q$. Assume that agents with preference type π^{ab} prefer a to b to $\emptyset$ to c to d, those with preference type π^{ba} prefer b to a to $\emptyset$ to c to d, those with preference type π^{cd} prefer c to d to $\emptyset$ to a to b, and those with preference type π^{dc} prefer d to c to $\emptyset$ to a to b.

For any q, the random assignments under RP^q for types π^{ab} and π^{ba} are

$$RP^q(\pi^{ab}) = (RP^q_a(\pi^{ab}), RP^q_b(\pi^{ab}), RP^q_c(\pi^{ab}), RP^q_d(\pi^{ab}), RP^q_\emptyset(\pi^{ab})) = \left(\frac{5}{12}, \frac{1}{12}, 0, 0, \frac{1}{2}\right),$$
$$RP^q(\pi^{ba}) = (RP^q_a(\pi^{ba}), RP^q_b(\pi^{ba}), RP^q_c(\pi^{ba}), RP^q_d(\pi^{ba}), RP^q_\emptyset(\pi^{ba})) = \left(\frac{1}{12}, \frac{5}{12}, 0, 0, \frac{1}{2}\right),$$

while the random assignments under PS^q are

$$PS^q(\pi^{ab}) = (PS^q_a(\pi^{ab}), PS^q_b(\pi^{ab}), PS^q_c(\pi^{ab}), PS^q_d(\pi^{ab}), PS^q_\emptyset(\pi^{ab})) = \left(\frac{1}{2}, 0, 0, 0, \frac{1}{2}\right),$$
$$PS^q(\pi^{ba}) = (PS^q_a(\pi^{ba}), PS^q_b(\pi^{ba}), PS^q_c(\pi^{ba}), PS^q_d(\pi^{ba}), PS^q_\emptyset(\pi^{ba})) = \left(0, \frac{1}{2}, 0, 0, \frac{1}{2}\right).$$

Therefore random priority and probabilistic serial do not converge to each other.

The above example shows that the two mechanisms do not necessarily converge to each other when the growth rates of different types of objects differ. However, the non-convergence seems to pose only a minor problem and have only limited influences on overall welfare. In Example 2, for instance, allocations for preference types π^{cd} and π^{dc} under RP and PS converge to each other as $q \to \infty$. Given that the proportions of agents of preference types π^{ab} and π^{ba} go to zero in this example, the inefficiency of RP still seems small in large economies.

7.4. **Multi-Unit Demands.** Consider a generalization of our basic setting in which each agent can obtain multiple units of objects. More specifically, we assume that there is a fixed integer k such that each agent can receive k objects. When $k = 1$, the model reduces to the model of the current paper. Assignment of popular courses in schools is one example of such a multiple unit assignment problem. See, for example, Kojima (2008) for formal definition of the model.

We consider two generalizations of the random priority mechanism to the current setting. In the **once-and-for-all random priority** mechanism, each agent i randomly draws a number f_i independently from a uniform distribution on $[0, 1]$ and, given the ordering, the agent with the lowest draw receives her favorite k objects, the agent with the second-lowest draw receives his favorite k objects from the remaining ones, and so forth. In the **draft random priority** mechanism, each agent i randomly draws a number f_i independently from a uniform distribution on $[0, 1]$. Second, the agent with the smallest draw receives her favorite object, the agent with the second-smallest draw receives his favorite object from the remaining ones, and so forth. Then agents obtain a random draw again and repeat the procedure k times.

We introduce two generalizations of the probabilistic serial mechanism. In the **multiunit-eating probabilistic serial** mechanism, each agent "eats" her k favorite available objects with speed one at every time $t \in [0, 1]$. In the **one-at-a-time probabilistic serial** mechanism, each agent "eats" the best available object with speed one at every time $t \in [0, k]$.

Our analysis can be adapted to this situation to show that the once-and-for-all random priority mechanism is asymptotically equivalent to the multiunit-eating probabilistic serial mechanism, whereas the draft random priority mechanism is asymptotically equivalent to the one-at-a-time probabilistic serial mechanism.

It is easy to see that the multiunit-eating probabilistic serial mechanism may not be ordinally efficient, while the one-at-a-time probabilistic serial mechanism is ordinally efficient. This may shed light on some issues in multiple unit assignment. It is well known that the once-and-for-all random priority mechanism is ex post efficient, but the mechanism is rarely used in practice. Rather, the draft mechanism is often used in application, for instance in sports drafting and allocations of courses in business schools. One of the reasons may be that the once-and-for-all random priority mechanism is ordinally inefficient even in the limit economy, whereas the draft random priority mechanism converges to an ordinally efficient mechanism as the economy becomes large — a reasonable assumption with course allocation in schools.

8. Concluding Remarks

Although the random priority (random serial dictatorship) mechanism is widely used for assigning objects to individuals, there has been increasing interest in the probabilistic serial mechanism as a potentially superior alternative. The tradeoffs associated with these mechanisms are multifaceted and difficult to evaluate in a finite economy. Yet, we have shown that the tradeoffs disappear, as the two mechanisms become effectively identical, in the large economy. More specifically, given a set of object types, the random assignments in these mechanisms converge to each other as the number of copies of each object type approaches infinity. This equivalence implies that the well-known concerns about the two mechanisms — the inefficiency of random priority and the incentive issue of probabilistic serial — abate in large markets.

Our result shares the recurring theme in economics that large economies can make things "right" in many settings. The benefits of large markets have been proven in many different circumstances, but no single insight appears to explain all of them, and one should not expect them to arise for all circumstances and for all mechanisms.

First, it is often the case that the large economy limits individuals' abilities and incentives to manipulate the mechanism. This is clearly the case for the Walrasian mechanism in exchange economy, as has been shown by Roberts and Postlewaite (1976). It is also the case for the deferred acceptance algorithm in two-sided matching (Kojima and Pathak (2008)) and for the probabilistic serial mechanism in one-sided matching (Kojima and Manea (2008)). Even this property is not to be taken for granted, however. The so-called **Boston mechanism** (Abdulkadiroğlu and Sönmez 2003b), which has been used to place students in public schools, provides an example. In that mechanism, a school first admits the students who rank it first, and if, and *only if*, there are seats left, admits those who rank it second, and so forth. It is well known that the students have incentives to misreport preferences in such a mechanism, and such manipulation incentives do not disappear as the economy becomes large.[31]

Second, one may expect that, with the diminished manipulation incentives, efficiency would be easier to obtain in a large economy. The asymptotic ordinal efficiency we find for the RP supports this impression. However, even some reasonable mechanisms fail to achieve asymptotic ordinal efficiency. Take the case of the **deferred acceptance algorithm with multiple tie-breaking (DA-MTB)**, an adaptation of the celebrated algorithm proposed by Gale and Shapley (1962) to the problem of assigning objects to agents, such as student assignment in public schools (see Abdulkadiroğlu, Pathak, and

[31]See Kojima and Pathak (2008) for a concrete example on this point.

Roth (2005)). In DA-MTB, *each object type* randomly and *independently* orders agents and, given the ordering, the assignment is decided by conducting the agent-proposing deferred acceptance algorithm with respect to the submitted preferences and the randomly decided priority profile. It turns out DA-MTB fails even ex post efficiency, let alone ordinal efficiency. Moreover, these inefficiencies do not disappear even in the continuum economy, as shown by Abdulkadiroğlu, Che, and Yasuda (2008).

Third, one plausible conjecture may be that the asymptotic ordinal efficiency is a necessary consequence of a mechanism that produces an ex post efficient assignment in every finite economy. This conjecture turns out to be false. Consider a family $\{\Gamma^q\}_{q\in\mathbb{N}}$ of replica economies and the following **replication-invariant random priority** mechanism $RIRP^q$. First, in the given q-economy, define a correspondence $\gamma : N^1 \twoheadrightarrow N^q$ such that $|\gamma(i)| = q$ for each $i \in N^1$, $\gamma(i) \cap \gamma(j) = \emptyset$ if $i \neq j$, and all agents in $\gamma(i)$ have the same preference as i. Call $\gamma(i)$ i's clones in the q-fold replica. Let each set $\gamma(i)$ of clones of agent i randomly draw a number f_i independently from a uniform distribution on $[0, 1]$. Second, all the clones with the smallest draw receive their favorite object, the clones with the second-smallest draw receive their most preferred object from the remaining ones, and so forth. This procedure induces a random assignment. It is clear that $RIRP^q = RP^1$ for any q-fold replica Γ^q. Therefore $||RIRP^q - RP^1|| \to 0$ as $q \to \infty$. Since RP^1 can be ordinally inefficient, the limit random assignment of $RIRP^q$ as $q \to \infty$ is not ordinally efficient in general.

Most importantly, our analysis shows the equivalence of two different mechanisms beyond showing certain asymptotic properties of given mechanisms. Such an equivalence is not expected even for a large economy, and has few analogues in the literature.

We conclude with possible directions of future research. First, little is known about matching and resource allocation in the face of aggregate uncertainty. This paper has made a first step in this direction, but a further study in designing mechanisms in such environments seems interesting. Second, we have studied a continuum economy model and provided its limit foundation. Continuum economies models are not yet common in the matching literature, so this methodology may prove useful more generally beyond the context of this paper. Finally, the random priority and the probabilistic serial mechanisms are equivalent only in the limit and do not exactly coincide in large but finite economies. How these competing mechanisms perform in finite economies remains an interesting open question.

Appendix

A. Proof of Theorem 1

It suffices to show that $\sup_{a\in O} |T_a^q - T_a^*| \to 0$ as $q \to \infty$. To this end, let

$$L > 2\max\left\{\max\{\tfrac{1}{m_a^*(O')}, m_a^*(O')\}\middle| O' \subset O, a \in O', m_a^*(O') > 0\right\}, \tag{A1}$$

and let $K := \min\{1 - x_a^*(v) \,|\, a \in O^*(v), v < \bar{v}^*\} > 0$, where $\bar{v}^* := \min\{v' | t^*(v') = 1\}$ is the last step of the recursive equations. Note (A1) implies $L > 2$.

Fix any $\epsilon > 0$ such that

$$2L^{4\bar{v}^*}\epsilon < \min\left\{K, \min_{v\in\{1,\dots,\bar{v}^*\}} |t^*(v) - t^*(v-1)|\right\}. \tag{A2}$$

By assumption there exists Q such that, for each $q > Q$,

$$|m_a^q(O') - m_a^\infty(O')| < \epsilon, \forall O' \subset \tilde{O}, \forall a \in O'. \tag{A3}$$

Fix any such q. For each $v \in \{1, ..., \bar{v}^*\}$, consider the set $A^*(v) := \{a \in O | T_a^* = t^*(v)\}$ of objects that expire at step v of PS^*. We show that $T_a^q \in (t^*(v) - L^{4v}\epsilon, t^*(v) + L^{4v}\epsilon)$ if and only if $a \in A^*(v)$. Let

$$J_v := \{i | t^q(i) = t_a^q(i) \text{ for some } a \in A^*(v)\}$$

be the steps at which the objects in $A^*(v)$ expire in PS^q. Clearly, it suffices to show that $t^q(i) \in (t^*(v) - L^{4v}\epsilon, t^*(v) + L^{4v}\epsilon)$ if and only if $i \in J_v$ We prove this recursively.

Suppose for each $v' \leq v-1$, $t^q(i') \in (t^*(v')-L^{4v'}\epsilon, t^*(v')+L^{4v'}\epsilon)$ if and only if $i' \in J_{v'}$, and further that, for each $a \in O^*(v-1)$, $x_a^q(k) \in (x_a^*(v-1)-L^{4(v-1)}\epsilon, x_a^*(v-1)+L^{4(v-1)}\epsilon)$, where k is the largest element of J_{v-1}. We shall then prove that $t^q(i) \in (t^*(v)-L^{4v}\epsilon, t^*(v)+L^{4v}\epsilon)$ if and only if $i \in J_v$, and that, for each $a \in O^*(v)$, $x_a^q(l) \in (x_a^*(v) - L^{4v}\epsilon, x_a^*(v) + L^{4v}\epsilon)$, where l is the largest element of J_v.

Observe first $O^q(k) = O^*(v-1)$, since k is the largest element of J_{v-1}.

Claim 1. For any $i > k$, $t^q(i) > t^*(v) - L^{4v-2}\epsilon$.

Proof. Suppose object $a \in O^*(v-1) = O^q(k)$ expires at step $k+1$ of PS^q. It suffices to show $t_a^q(k+1) > t^*(v) - L^{4v-2}\epsilon$. Suppose to the contrary that

$$t_a^q(k+1) \leq t^*(v) - L^{4v-2}\epsilon. \tag{A4}$$

Recall, by the inductive assumption, that

$$x_a^q(k) < x_a^*(v-1) + L^{4(v-1)}\epsilon. \tag{A5}$$

Thus,

$$\begin{aligned}
x_a^q(k+1) &= x_a^q(k) + m_a^q(O^q(k))(t_a^q(k+1) - t^q(k)) \\
&\leq x_a^q(k) + m_a^q(O^q(k))(t^*(v) - L^{4v-2}\epsilon - t^*(v-1) + L^{4(v-1)}\epsilon) \\
&\leq x_a^q(k) + m_a^q(O^q(k))[t^*(v) - t^*(v-1) - L^{4v-3}\epsilon] \\
&< x_a^*(v-1) + L^{4(v-1)}\epsilon + m_a^\infty(O^*(v-1))[t^*(v) - t^*(v-1) - L^{4v-3}\epsilon] + \epsilon, \quad \text{(A6)}
\end{aligned}$$

where the first equality follows from definition of PS^q (4.4) and the fact that $t_a^q(k+1) = t^q(k+1)$, the first inequality follows from the inductive assumption and (A4), the second inequality holds since $L^{4v-2}\epsilon - L^{4(v-1)}\epsilon = L^{4v-3}(L - \frac{1}{L})\epsilon > L^{4v-3}\epsilon$ since $L > 2$, which follows from (A1), and the third inequality follows from (A2), (A3) and (A5).[32]

There are two cases. Suppose first $m_a^\infty(O^*(v-1)) = 0$. Then, the last line of (A6) becomes

$$x_a^*(v-1) + L^{4(v-1)}\epsilon + \epsilon,$$

which is strictly less than 1, by $a \in O^*(v-1)$ and (A2). Suppose next $m_a^\infty(O^*(v-1)) > 0$. Then, the last line of (A6) equals

$$\begin{aligned}
& x_a^*(v-1) + L^{4(v-1)}\epsilon + m_a^\infty(O^*(v-1))[t^*(v) - t^*(v-1) - L^{4v-3}\epsilon] + \epsilon \\
< \;& x_a^*(v-1) + m_a^\infty(O^*(v-1))[t^*(v) - t^*(v-1)] \\
\leq \;& 1,
\end{aligned}$$

where the first inequality holds since, by (A1), $m_a^\infty(O^*(v-1))L^{4v-3}\epsilon > 2L^{4(v-1)}\epsilon \geq L^{4(v-1)}\epsilon + \epsilon$, and the second follows since $a \in O^*(v-1)$. In either case, we have a contradiction to the fact that a expires at step $k+1$. ‖

Claim 2. For any $i \in J_v$, then $t^q(i) \leq t^*(v) + L^{4v-2}\epsilon$.

Proof. Suppose a expires at step $l \equiv \max J_v$ of PS^q. It suffices to show $t^q(l) = t_a^q(l) \leq t^*(v) + L^{4v-2}\epsilon$. If $t^*(v) = 1$, then this is trivially true. Thus, let us assume $t_a^*(v) < 1$. This implies $m_a^\infty(O^*(v-1)) > 0$. For that case, suppose for contradiction that

$$t_a^q(l) > t^*(v) + L^{4v-2}\epsilon. \quad \text{(A7)}$$

[32] By (A2), $t^*(v) - t^*(v-1) - L^{4v-3}\epsilon \in (0,1)$, so

$$\begin{aligned}
& m_a^\infty(O^*(v-1))[t^*(v) - t^*(v-1) - L^{4v-3}\epsilon] - m_a^q(O^q(k))[t^*(v) - t^*(v-1) - L^{4v-3}\epsilon] \\
= \;& (m_a^\infty(O^*(v-1)) - m_a^q(O^q(k)))[t^*(v) - t^*(v-1) - L^{4v-3}\epsilon] < m_a^\infty(O^*(v-1)) - m_a^q(O^q(k)) < \epsilon,
\end{aligned}$$

where the last inequality follows from (A3).

Then,

$$\begin{aligned}
x_a^q(l) &= x_a^q(k) + \sum_{j=k+1}^{l} m_a^q(O^q(j-1))[t^q(j) - t^q(j-1)] \\
&\geq x_a^q(k) + \sum_{j=k+1}^{l} m_a^q(O^q(k))[t^q(j) - t^q(j-1)] \\
&= x_a^q(k) + m_a^q(O^*(v-1))[t^q(l) - t^q(k)] \\
&> x_a^*(v-1) - L^{4(v-1)}\epsilon + m_a^q(O^*(v-1))[t^*(v) + L^{4v-2}\epsilon - t^*(v-1) - L^{4(v-1)}\epsilon] \\
&\geq x_a^*(v-1) - L^{4(v-1)}\epsilon + m_a^\infty(O^*(v-1))[t^*(v) - t^*(v-1) + L^{4v-3}\epsilon] \\
&> x_a^*(v-1) + m_a^\infty(O^*(v-1))[t^*(v) - t^*(v-1)] \\
&= x_a^*(v) = 1,
\end{aligned}$$

where the first equality follows from (4.4), the first inequality follows since $m_a^q(O^q(j-1)) \geq m_a^q(O^q(k))$ for each $j \geq k+1$ by $O^q(j-1) \subseteq O^q(k)$, the second equality from $O^q(k) = O^*(v-1)$, the second inequality follows from the inductive assumption and (A7), the third inequality follows from the assumption (A1), and the fourth inequality follows from (A1) and $m_a^\infty(O^*(v-1)) > 0$. Thus $x_a^q(l) > 1$, which contradicts the definition of $x_a^q(l)$. ‖

Claim 3. If $i \in J_{v'}$ for some $v' > v$, then $t^q(i) > t^*(v) + L^{4v}\epsilon$.

Proof. Suppose otherwise. Let c be the object that expires the first among $O^*(v)$ in PS^q. Let j be the step at which it expires. We must have

$$t^q(j) \leq t^*(v) + L^{4v}\epsilon. \tag{A8}$$

In particular, $t_c^q(j) < 1$ and $x_c^q(j) = 1$. Since c is the first object to expire in $O^*(v)$, at each of steps $k+1, \ldots, j-1$, some object in $A^*(v)$ expires. (If $j = k+1$, then no other object expires in between step k and step j.) Also, by Claim 1,

$$t^q(k+1) > t^*(v) - L^{4v-2}\epsilon. \tag{A9}$$

Therefore,

$$\begin{aligned} x_c^q(j) &= x_c^q(k) + \sum_{i=k+1}^{j} m_c^q(O^q(i-1))(t^q(i) - t^q(i-1)) \\ &\leq x_c^q(k) + m_c^q(O^q(k))(t^q(k+1) - t^q(k)) + m_c^q(O^q(j-1))(t^q(j) - t^q(k+1)) \\ &\leq x_c^*(v-1) + L^{4(v-1)}\epsilon + (m_c^*(O^q(k)) + \epsilon)\left((t^*(v) + L^{4v-2}\epsilon) - (t^*(v-1) - L^{4(v-1)}\epsilon)\right) \\ &\qquad + (m_a^*(O^q(j)) + \epsilon)\left(L^{4v}\epsilon - L^{4v-2}\epsilon\right) \\ &\leq x_c^*(v) + L^{4v+1}\epsilon \\ &\leq 1 - K + L^{4\bar{v}^*}\epsilon \\ &< 1, \end{aligned}$$

where the first equality follows from (4.4), the first inequality follows since $m_c^q(O^q(i-1)) \leq m_c^q(O^q(j-1))$ for any $i \leq j$ by $O^q(i-1) \subset O^q(j-1)$, the second inequality follows from the inductive assumption, (A3), (A9), and (A8), the third inequality follows from (A1), and the last inequality follows from (A2) and the definition of K. This contradicts the assumption that c expires at step j. ‖

Claims 1-3 prove that $t^q(i) \in (t^*(v) - L^{4v-2}\epsilon, t^*(v) + L^{4v-2}\epsilon) \subset (t^*(v) - L^{4v}\epsilon, t^*(v) + L^{4v}\epsilon)$ if and only if $i \in J_v$, which in turn implies that $T_a^q \in (t^*(v) - L^{4v}\epsilon, t^*(v) + L^{4v}\epsilon)$ if and only if $a \in A^*(v)$. It now remains to prove the following:

Claim 4. For each $a \in O^*(v)$, $x_a^q(l) \in (x_a^*(v) - L^{4v}\epsilon, x_a^*(v) + L^{4v}\epsilon)$, where l is the largest element of J_v.

Proof. Fix any $a \in O^*(v)$. Then,

$$\begin{aligned} x_a^q(l) &= x_a^q(k) + \sum_{j=k+1}^{l} m_a^q(O^q(j-1))(t^q(j) - t^q(j-1)) \\ &\leq x_a^q(k) + m_a^q(O^q(k))(t^q(k+1) - t^q(k)) + m_a^q(O^q(l-1))(t^q(l) - t^q(k+1)) \\ &\leq x_a^*(v-1) + L^{4(v-1)}\epsilon + (m_a^*(O^q(k)) + \epsilon)\left(t^*(v) - t^*(v-1) + 2L^{4v-2}\epsilon\right) \\ &\qquad + (m_a^*(O^q(l-1)) + \epsilon)\left(2L^{4v-2}\epsilon\right) \\ &< x_a^*(v-1) + m_a^*(O^*(v-1))(t^*(v) - t^*(v-1)) + L^{4v}\epsilon \\ &= x_a^*(v) + L^{4v}\epsilon, \end{aligned}$$

where the first equality follows from (4.4), the first inequality follows since $m_c^q(O^q(i-1)) \leq m_c^q(O^q(l-1))$ for any $i \leq l$ by $O^q(i-1) \subset O^q(l-1)$, the second inequality follows from the inductive assumption, (A3), Claims 1 and 2, the third inequality follows from (A1), and the last equality follows from (5.4).

A symmetric argument yields $x_a^q(l) \geq x_a^*(v) - L^{4v}\epsilon$. ∥

We have thus completed the recursive argument, which taken together proves that $T_a^q \in (t^*(v) - L^{4v}\epsilon, t^*(v) + L^{4v}\epsilon)$ if and only if $t_a^*(v) = t^*(v)$, for any $q > Q$ for some $Q \in \mathbb{N}$. Since $\epsilon > 0$ can be arbitrarily small, $T_a^q \to T_a^*$ as $q \to \infty$. Since there are only a finite number of objects and a finite number of preference types, $||PS^q - PS^*|| \to 0$ as $q \to \infty$.

B. Proof of Theorem 2

As with the proof of Theorem 1, let L be a real number satisfying condition (A1) and let $K := \min\{1 - x_a^*(v) \,|\, a \in O^*(v), v < \bar{v}^*\} > 0$, where $\bar{v}^* := \min\{v' | t^*(v') = 1\}$ is the last step of the recursive equations.

Fix an agent i_0 of preference type $\pi_0 \in \Pi$ and consider the random assignment for agents of type π_0. Consider the following events:

$$E_1^q(\pi) : \hat{m}_\pi^q(t^*(v-1) - L^{4(v-1)}\epsilon, t^*(v) - L^{4v-2}\epsilon) < m_\pi^\infty[t^*(v) - t^*(v-1) - L^{4v-3}\epsilon], \text{ for all } v,$$

$$E_2^q(\pi) : \hat{m}_\pi^q(t^*(v-1) + L^{4(v-1)}\epsilon, t^*(v) + L^{4v-2}\epsilon) \geq m_\pi^\infty[t^*(v) - t^*(v-1) + L^{4v-3}\epsilon], \text{ for all } v \neq \bar{v}^*,$$

$$E_3^q(\pi) : \hat{m}_\pi^q(t^*(v-1) - L^{4(v-1)}\epsilon, t^*(v) + L^{4v-2}\epsilon) < m_\pi^\infty[t^*(v) - t^*(v-1) + 2L^{4v-2}\epsilon], \text{ for all } v,$$

$$E_4^q(\pi) : \hat{m}_\pi^q(t^*(v) - L^{4v-2}\epsilon, t^*(v) + L^{4v}\epsilon) < m_\pi^\infty \times 2L^{4v}\epsilon, \text{ for all } v,$$

$$E_5^q(\pi) : \hat{m}_\pi^q(t^*(v) - L^{4v-2}\epsilon, t^*(v) + L^{4v-2}\epsilon) < m_\pi^\infty \times 3L^{4v-2}\epsilon, \text{ for all } v,$$

$$E_6^q(\pi) : \hat{m}_\pi^q(t^*(v-1) + L^{4(v-1)}\epsilon, t^*(v) - L^{4v-2}\epsilon) \geq m_\pi^\infty[t^*(v) - t^*(v-1) - 2L^{4v-2}\epsilon] \text{ for all } v.$$

Before presenting a formal proof of Theorem 2, we describe its outline. First, Lemma 1 below shows that all the cutoff times of RP^q become arbitrarily close to the corresponding expiration dates of PS^* as $q \to \infty$ when event $E_i^q(\pi)$ holds for every π and $i \in \{1, \ldots, 6\}$. Then, in the proof of Theorem 2, (1) we use Lemma 1 to show that the conditional probability of obtaining an object under RP^q is close to the probability of receiving that object under PS^*, given all the events of the form $E_i^q(\pi)$; and (2) we show that the probability that all the events of the form $E_i^q(\pi)$ hold approaches one as q goes to infinity, so the overall, unconditional probability of obtaining each object in RP^q is close to the conditional probability of receiving that object, given all the events of the form $E_i^q(\pi)$. We finally complete the proof of the Theorem by combining items (1) and (2) above.

Lemma 1. For any $\epsilon > 0$ such that

$$\text{(B1)} \qquad 2L^{4\bar{v}^*}\epsilon < \min\left\{\min_{v \in \{1,\ldots,\bar{v}^*\}}\{t^*(v) - t^*(v-1)\}, K\right\},$$

there exists Q such that the following is true for any $q > Q$: if the realization of $f_{-i_0} \in [0,1]^{|N^q|-1}$ is such that events $E_1^q(\pi)$, $E_2^q(\pi)$, $E_3^q(\pi)$, $E_4^q(\pi)$, $E_5^q(\pi)$, and $E_6^q(\pi)$ hold for all $\pi \in \Pi$ with $m_\pi^\infty > 0$, then $\hat{T}_a^q \in (t^*(v) - L^{4v}\epsilon, t^*(v) + L^{4v}\epsilon)$ if and only if $t_a^*(v) = t^*(v)$.

Before presenting a complete proof of Lemma 1, we note that the proof closely follows the proof of Theorem 1. More specifically, the proof of Theorem 1 shows inductively that the expiration date of each object type in PS^q is close to that of PS^* when q is large enough, while the proof of Lemma 1 shows inductively that the cutoff time of each object type in RP^q is close to that of PS^* when all the events of the form $E_i^q(\pi)$ hold. Indeed, Claims 1, 2, 3, and 4 in the proof of Theorem 1 correspond to Claims 5, 6, 7, and 8 in the proof of Lemma 1, respectively. Both arguments utilize the fact that the average rates of consumption of each object type in PS^q and RP^q are close to those under PS^* during relevant time intervals. The main difference between the proofs of Theorem 1 and Lemma 1 is the following: consumption rates of PS^q are close to PS^* because m_π^q is close to m_π^∞ for all a and π when q is large, whereas consumption rates of RP^q are *assumed* to be close by all the events of the form $E_i^q(\pi)$, and Lemma 1 shows that these events in fact make the cutoff times in RP^q close to expiration dates in PS^*. As mentioned above, the proof of Theorem 2 then shows that assuming all the events of the form $E_i^q(\pi)$ is not problematic, since the probability of these events converges to one as q approaches infinity.

Proof of Lemma 1. There exists Q such that

(B2) $$\sum_{\pi \in \Pi : m_\pi^\infty = 0} m_\pi^q < \epsilon,$$

for any $q > Q$. Fix any such q and suppose that the realization of f_{-i_0} is such that $E_1^q(\pi)$, $E_2^q(\pi)$, $E_3^q(\pi)$, $E_4^q(\pi)$, $E_5^q(\pi)$, and $E_6^q(\pi)$ hold for all π with $m_\pi^\infty > 0$ as described in the statement of the Lemma. We first define the steps

$$\hat{J}_v := \{i | \hat{t}_a^q(i) = \hat{t}^q(i) \text{ for some } a \in A^*(v)\}$$

at which the objects in $A^*(v)$ expire in RP^q. The lemma shall be proven by showing that $\hat{t}^q(i) \in (t^*(v) - L^{4v}\epsilon, t^*(v) + L^{4v}\epsilon)$ if and only if $i \in \hat{J}_v$. We show this inductively.

Suppose for any $v' \leq v-1$, $\hat{t}^q(i') \in (t^*(v') - L^{4v'}\epsilon, t^*(v') + L^{4v'}\epsilon)$ if and only if $i' \in \hat{J}_{v'}$, and further that, for each $a \in O^*(v-1)$, $\hat{x}_a^q(k) \in (x_a^*(v-1) - L^{4(v-1)}\epsilon, x_a^*(v-1) + L^{4(v-1)}\epsilon)$, where k is the largest element of $\hat{J}_{v-1}$. We shall then prove that $\hat{t}^q(i) \in (t^*(v) - L^{4v}\epsilon, t^*(v) + L^{4v}\epsilon)$ if and only if $i \in \hat{J}_v$, and that, for each $a \in O^*(v)$, $\hat{x}_a^q(l) \in (x_a^*(v) - L^{4v}\epsilon, x_a^*(v) + L^{4v}\epsilon)$, where l is the largest element of $\hat{J}_v$.

Let k be the largest element of $\hat{J}_{v-1}$. It then follows that $\hat{O}^q(k) = O^*(v-1)$.

Claim 5. For any $i > k$, $\hat{t}^q(i) > t^*(v) - L^{4v-2}\epsilon$.

Proof. Suppose object $a \in O^*(v-1) = O^q(k)$ expires at step $k+1$ of RP^q. It suffices to show $\hat{t}^q_a(k+1) > t^*(v) - L^{4v-2}\epsilon$. Suppose to the contrary that

$$\hat{t}^q_a(k+1) \leq t^*(v) - L^{4v-2}\epsilon. \tag{B3}$$

Recall, by inductive assumption, that

$$\hat{x}^q_a(k) < x^*_a(v-1) + L^{4(v-1)}\epsilon. \tag{B4}$$

Thus,

$$\begin{aligned}
\hat{x}^q_a(k+1) &= \hat{x}^q_a(k) + \hat{m}^q_a(\hat{O}^q(k); \hat{t}^q(k), \hat{t}^q_a(k+1)) \\
&\leq \hat{x}^q_a(k) + \hat{m}^q_a(\hat{O}^q(k); t^*(v-1) - L^{4(v-1)}\epsilon, t^*(v) - L^{4v-2}\epsilon) \\
&< x^*_a(v-1) + L^{4(v-1)}\epsilon + m^\infty_a(O^*(v-1))[t^*(v) - t^*(v-1) - L^{4v-3}\epsilon] + \epsilon, \qquad \text{(B5)}
\end{aligned}$$

where the first equality follows from (4.8) in the definition of RP^q, the first inequality follows from the inductive assumption and (B3), and the second inequality follows from the assumption that $E^q_1(\pi)$ holds for all $\pi \in \Pi$ and conditions (B2) and (B4).

There are two cases. Suppose first $m^\infty_a(O^*(v-1)) = 0$. Then, the last line of (B5) becomes

$$x^*_a(v-1) + L^{4(v-1)}\epsilon + \epsilon,$$

which is strictly less than 1, since $a \in O^*(v-1)$ and since (B1) holds. Suppose next $m^\infty_a(O^*(v-1)) > 0$. Then, the last line of (B5) equals

$$\begin{aligned}
& x^*_a(v-1) + L^{4(v-1)}\epsilon + m^\infty_a(O^*(v-1))[t^*(v) - t^*(v-1) - L^{4v-3}\epsilon] + \epsilon \\
< \;& x^*_a(v-1) + m^\infty_a(O^*(v-1))[t^*(v) - t^*(v-1)] \\
\leq \;& 1,
\end{aligned}$$

where the first inequality follows from (A1), and the second follows since $a \in O^*(v-1)$. In either case, we have a contradiction to the fact that a expires at step $k+1$. ‖

Claim 6. For any $i \in \hat{J}_v$, then $\hat{t}^q(i) \leq t^*(v) + L^{4v-2}\epsilon$.

Proof. Suppose a expires at step $l \equiv \max \hat{J}_v$ of RP^q. It suffices to show $\hat{t}^q(l) = \hat{t}^q_a(l) \leq t^*(v) + L^{4v-2}\epsilon$. If $t^*(v) = 1$, then the claim is trivially true. Thus, let us assume $t^*(v) < 1$. This implies $m^\infty_a(O^*(v-1)) > 0$. For that case suppose, for contradiction, that

$$\hat{t}^q(l) > t^*(v) + L^{4v-2}\epsilon. \tag{B6}$$

Then,

$$
\begin{aligned}
\hat{x}^q_a(l) &= \hat{x}^q_a(k) + \sum_{j=k+1}^{l} \hat{m}^q_a(\hat{O}^q(j-1); \hat{t}^q(j-1), \hat{t}^q(j)) \\
&\geq \hat{x}^q_a(k) + \sum_{j=k+1}^{l} \hat{m}^q_a(\hat{O}^q(k); \hat{t}^q(j-1), \hat{t}^q(j)) \\
&= \hat{x}^q_a(k) + \hat{m}^q_a(O^*(v-1); \hat{t}^q(k), \hat{t}^q(l)) \\
&> x^*_a(v-1) - L^{4(v-1)}\epsilon + \hat{m}^q_a(O^*(v-1); t^*(v-1) + L^{4(v-1)}\epsilon, t^*(v) + L^{4v-2}\epsilon) \\
&\geq x^*_a(v-1) - L^{4(v-1)}\epsilon + m^\infty_a(O^*(v-1))[t^*(v) - t^*(v-1) + L^{4v-3}\epsilon] \\
&> x^*_a(v-1) + m^\infty_a(O^*(v-1))[t^*(v) - t^*(v-1)] \\
&= x^*_a(v) = 1,
\end{aligned}
$$

where the first equality follows from (4.8), the first inequality follows since $\hat{m}^q_a(\hat{O}^q(j-1); t, t') \geq m^q_a(\hat{O}^q(k); t, t')$ for any $j \geq k+1$ and $t \leq t'$ by $\hat{O}^q(j-1) \subseteq \hat{O}^q(k)$, the second equality from $\hat{O}^q(k) = O^*(v-1)$ and the definition of $\hat{m}^q_a$, the second inequality follows from the inductive assumption and (B6), the third inequality follows from the assumption that $E^q_2(\pi)$ holds, and the fourth inequality follows from (A1) and the assumption $m^\infty_a(O^*(v-1)) > 0$. Thus $\hat{x}^q_a(l) > 1$, which contradicts the definition of $x^q_a(l)$. ‖

Claim 7. If $i \in \hat{J}_{v'}$ for some $v' > v$, then $\hat{t}^q(i) > t^*(v) + L^{4v}\epsilon$.

Proof. Suppose otherwise. Let c be the object that expires the first among $O^*(v)$ in RP^q. Let j be the step at which it expires. Then, we must have

$$\hat{t}^q_c(j) \leq t^*(v) + L^{4v}\epsilon, \tag{B7}$$

and $\hat{x}^q_c(j) = 1$. Since c is the first object to expire in $O^*(v)$, at each of steps $k+1, \ldots, j-1$, some object in $A^*(v)$ expires. (If $j = k+1$, then no other object expires in between step

k and step j.) By Claim 5, this implies $\hat{t}^q(k+1) > t^*(v) - L^{4v-2}\epsilon$. Therefore,

$$\begin{aligned}
\hat{x}^q_c(j) &= \hat{x}^q_c(k) + \sum_{i=k+1}^{j} \hat{m}^q_c(\hat{O}^q(i-1); \hat{t}^q(i-1), \hat{t}^q(i)) \\
&\leq \hat{x}^q_c(k) + \hat{m}^q_c(\hat{O}^q(k); \hat{t}^q(k), \hat{t}^q(k+1)) + \hat{m}^q_c(\hat{O}^q(j-1); \hat{t}^q(k+1), \hat{t}^q(j)) \\
&\leq \hat{x}^q_c(k) + \hat{m}^q_c(\hat{O}^q(k); t^*(v-1) - L^{4(v-1)}\epsilon, t^*(v) + L^{4v-2}\epsilon) \\
&\quad + \hat{m}^q_c(\hat{O}^q(j-1); t^*(v) - L^{4v-2}\epsilon, t^*(v) + L^{4v}\epsilon) \\
&\leq x^*_c(v-1) + L^{4(v-1)}\epsilon + m^*_c(O^*(v-1))[t^*(v) - t^*(v-1) + 2L^{4v-2}\epsilon] \\
&\quad + m^\infty_c(\hat{O}^q(j-1)) \times 2L^{4v}\epsilon + \epsilon \\
&\leq x^*_c(v) + L^{4v+1}\epsilon \\
&\leq 1 - K + L^{4\bar{v}^*}\epsilon \\
&< 1,
\end{aligned}$$

where the first equality follows from (4.8), the first inequality follows since $\hat{m}^q_c(\hat{O}^q(j-1); t, t') \geq m^q_c(\hat{O}^q(i-1); t, t')$ for any $j \geq i$ by $\hat{O}^q(j-1) \subseteq \hat{O}^q(i-1)$, the second inequality follows from the inductive assumption, and Claims 5 and 6, the third inequality follows from the inductive assumption, $E^q_3(\pi)$, $E^q_4(\pi)$ and (B2), the fourth inequality follows from (5.4) and (A1), the fifth inequality follows from the definition of K, and the last inequality follows from the assumption that $2L^{4\bar{v}^*}\epsilon < K$. Thus we obtain $\hat{x}^q_c(j) < 1$, which contradicts the assumption that c expires at step j. ‖

Claims 5, 6, and 7 prove that $\hat{t}^q(i) \in (t^*(v) - L^{4v-2}\epsilon, t^*(v) + L^{4v-2}\epsilon) \subset (t^*(v) - L^{4v}\epsilon, t^*(v) + L^{4v}\epsilon)$ if and only if $i \in \hat{J}_v$. This implies that $\hat{T}^q_a \in (t^*(v) - L^{4v}\epsilon, t^*(v) + L^{4v}\epsilon)$ if and only if $a \in A^*(v)$. It now remains to show the following.

Claim 8. For each $a \in O^*(v)$, $x^q_a(l) \in (x^*_a(v) - L^{4v}\epsilon, x^*_a(v) + L^{4v}\epsilon)$, where l is the largest element of $\hat{J}_v$.

Proof. Fix any $a \in O^*(v)$. Then,

$$\begin{aligned}
\hat{x}^q_a(l) &= \hat{x}^q_a(k) + \sum_{j=k+1}^{l} \hat{m}^q_a(\hat{O}^q(j-1); \hat{t}^q(j-1), \hat{t}^q(j)) \\
&\leq \hat{x}^q_a(k) + \hat{m}^q_a(\hat{O}^q(k); \hat{t}^q(k), \hat{t}^q(k+1)) + \hat{m}^q_a(\hat{O}^q(l); \hat{t}^q(k+1), \hat{t}^q(l)) \\
&\leq \hat{x}^q_a(k) + \hat{m}^q_a(\hat{O}^q(k); t^*(v-1) - L^{4(v-1)}\epsilon, t^*(v) + L^{4v-2}\epsilon) \\
&\qquad + \hat{m}^q_a(\hat{O}^q(l); t^*(v) - L^{4v-2}\epsilon, t^*(v) + L^{4v-2}\epsilon) \\
&< x^*_a(v-1) + L^{4(v-1)}\epsilon + m^*_a(\hat{O}^q(k))(t^*(v) - t^*(v-1) + 2L^{4v-2}\epsilon) \\
&\qquad + m^*_a(\hat{O}^q(l)) \times 3L^{4v-2}\epsilon + 2\epsilon \\
&< x^*_a(v-1) + (m^*_a(O^*(v-1)))\,(t^*(v) - t^*(v-1)) + L^{4v}\epsilon \\
&= x^*_a(v) + L^{4v}\epsilon,
\end{aligned}$$

where the first equality follows from (4.8), the first inequality follows from $m^q_a(\hat{O}^q(l); t, t') \geq m^q_a(\hat{O}^q(j); t, t')$ for all $l \geq j$, the second inequality follows from the inductive assumption and Claims 5 and 6, the third inequality follows from the inductive assumption, (B2) and $E^q_3(\pi)$ and $E^q_5(\pi)$, the fourth inequality follows from $\hat{O}^q(k) = O^*(v-1)$ and (A1), and the last inequality follows from (5.4).

Next we obtain

$$\begin{aligned}
\hat{x}^q_a(l) &= \hat{x}^q_a(k) + \sum_{j=k+1}^{l} \hat{m}^q_a(\hat{O}^q(j-1); \hat{t}^q(j-1), \hat{t}^q(j)) \\
&\geq \hat{x}^q_a(k) + \hat{m}^q_a(\hat{O}^q(k); \hat{t}^q(k), \hat{t}^q(l)) \\
&\geq \hat{x}^q_a(k) + \hat{m}^q_a(\hat{O}^q(k); t^*(v-1) + L^{4(v-1)}\epsilon, t^*(v) - L^{4v-2}\epsilon) \\
&\geq x^*_a(v-1) - L^{4(v-1)}\epsilon + m^*_a(O^*(v-1))[t^*(v) - t^*(v-1) - 2L^{4v-2}\epsilon] \\
&> x^*_a(v) - L^{4v}\epsilon,
\end{aligned}$$

where the first inequality follows from $\hat{O}^q(j-1) \subseteq \hat{O}^q(k)$ for any $j \geq k+1$, the second inequality follows from the inductive assumption and Claim 5, the third inequality follows from the inductive assumption and $E^q_6(\pi)$, and the last inequality follows from (5.4) and (A1). These inequalities complete the proof. ‖

We have thus completed the recursive argument, which taken together proves that $\hat{T}^q_a \in (t^*(v) - L^{4v}\epsilon, t^*(v) + L^{4v}\epsilon)$ if and only if $a \in A^*(v)$, for any $q > Q$ for some $Q \in \mathbb{N}$. □

Proof of Theorem 2. We shall show that for any $\varepsilon > 0$ there exists Q such that, for any $q > Q$, for any $\pi_0 \in \Pi$ and $a \in O$,

$$|PS_a^*(\pi_0) - RP_a^q(\pi_0)| < (2L^{4(n+1)} + 6(n+1)!)\varepsilon. \tag{B8}$$

Since n is a finite constant, relation (B8) implies the Theorem.

To show this, first assume without loss of generality that ϵ satisfies (B1) and Q is so large that (B2) holds for any $q > Q$. We have

$$\begin{aligned}
RP_a^q(\pi_0) &= \mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0)\right] \\
&= \mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0) \middle| \bigcap_{i=1}^{6} \bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)\right] \times Pr\left[\bigcap_{i=1}^{6} \bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)\right] \\
&+ \mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0) \middle| \overline{\bigcap_{i=1}^{6} \bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)}\right] \times Pr\left[\overline{\bigcap_{i=1}^{6} \bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)}\right] \\
&= \mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0) \middle| \bigcap_{i=1}^{6} \bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)\right] \times \left(1 - Pr\left[\bigcup_{i=1}^{6} \bigcup_{\pi\in\Pi:m_\pi^\infty>0} \overline{E_i^q(\pi)}\right]\right) \\
&+ \mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0) \middle| \overline{\bigcap_{i=1}^{6} \bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)}\right] \times Pr\left[\bigcup_{i=1}^{6} \bigcup_{\pi\in\Pi:m_\pi^\infty>0} \overline{E_i^q(\pi)}\right] \\
&= \mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0) \middle| \bigcap_{i=1}^{6} \bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)\right] \\
&+ \left\{\mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0) \middle| \overline{\bigcap_{i=1}^{6} \bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)}\right] - \mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0) \middle| \bigcap_{i=1}^{6} \bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)\right]\right\} \\
&\times Pr\left[\bigcup_{i=1}^{6} \bigcup_{\pi\in\Pi:m_\pi^\infty>0} \overline{E_i^q(\pi)}\right],
\end{aligned} \tag{B9}$$

where for any event E, $\mathbb{E}[\cdot|E]$ denotes the conditional expectation given E, and $\bar{E}$ is the complement event of E.

First, we bound the first term of expression (B9). Since $\bar{v}^* \leq n+1$, Lemma 1 implies that

$$\mathbb{E}\left[\hat{T}_a^q - \hat{\tau}_a^q(\pi_0) \middle| \bigcap_{i=1}^{6} \bigcap_{\pi\in\Pi:m_\pi^\infty>0} E_i^q(\pi)\right] \in [T_a^\infty - \tau_a^*(\pi_0) - 2L^{4(n+1)}\epsilon, T_a^\infty - \tau_a^*(\pi_0) + 2L^{4(n+1)}\epsilon].$$

Second, we bound the second term of expression (B9). By the weak law of large numbers, for any $\varepsilon > 0$, there exists Q such that $Pr\left[\overline{E_i^q(\pi)}\right] < \epsilon$ for any $i \in \{1,2,3,4,5,6\}$, $q > Q$ and $\pi \in \Pi$ with $m_\pi^\infty > 0$. Since there are at most $6(n+1)!$ such events and, in general, the sum of probabilities of a number of events is weakly larger than the probability of the union of the events (Boole's inequality), we obtain

$$\begin{aligned} Pr\left[\bigcup_{i=1}^{6} \bigcup_{\pi \in \Pi : m_\pi^\infty > 0} \overline{E_i^q(\pi)}\right] &\leq \sum_{i=1}^{6} \sum_{\pi \in \Pi : m_\pi^\infty > 0} Pr\left[\overline{E_i^q(\pi)}\right] \\ &\leq 6(n+1)!\epsilon. \end{aligned}$$

Since $\hat{T}_a^q - \hat{\tau}_a^q(\pi_0) \in [0,1]$ for any a, q and π_0, the second term of equation (B9) is in $[-6(n+1)!\epsilon, 6(n+1)!\epsilon]$.

From the above arguments and the definition $PS_a^*(\pi_0) = T_a^\infty - \tau_a^*(\pi_0)$ for every a and π_0, we have that

$$|PS_a^*(\pi_0) - RP_a^q(\pi_0)| < (2L^{4(n+1)} + 6(n+1)!)\epsilon,$$

completing the proof. □

C. Proof of Proposition 3

The proposition uses the following two lemmas. Let $\{\Gamma^q\}$ be a family of replica economies. Given any q, define a correspondence $\gamma : N^1 \twoheadrightarrow N^q$ such that $|\gamma(i)| = q$ for each $i \in N^1$, $\gamma(i) \cap \gamma(j) = \emptyset$ if $i \neq j$, and all agents in $\gamma(i)$ have the same preference as i. Call $\gamma(i)$ i's clones in the q-fold replica.

Lemma 2. For all $q \in \mathbb{N}$ and $a, b \in \tilde{O}$, $a \rhd (RP^1, m^1)\, b \iff a \rhd (RP^q, m^q)\, b$.

Proof. We proceed in two steps.

(i) $a \rhd (RP^1, m^1)\, b \Longrightarrow a \rhd (RP^q, m^q)\, b$: Suppose first $a \rhd (RP^1, m^1)\, b$. There exists an individual $i^* \in N^1$ and an ordering $(i^1_{(1)}, \ldots, i^1_{(|N^1|)})$ (implied by some draw $f^1 \in [0,1]^{|N^1|}$) such that the agents in front of i^* in that ordering consume all the objects that i^* prefers to b but not b, and i^* consumes b.

Now consider the q-fold replica. With positive probability, we have an ordering $(\bar{\gamma}(i^1_{(1)}), \ldots, \bar{\gamma}(i^1_{(|N^1|)}))$, where $\bar{\gamma}(i)$ is an arbitrary permutation of $\gamma(i)$. Under this ordering, each agent in $\gamma(i^1_{(j)})$ will consume a copy of the object agent $i^1_{(j)}$ will consume in the base economy, and hence all the agents in $\gamma(i^*)$ will consume b (despite preferring a to b). This proves that $a \rhd (RP^q, m^q)\, b$.

(ii) $a \rhd (RP^q, m^q)\, b \Longrightarrow a \rhd (RP^1, m^1)\, b$: Suppose $a \rhd (RP^q, m^q)\, b$. Then, with positive probability, a draw $f^q \in [0,1]^{|N^q|}$ entails an ordering in which the agents ahead of $i^* \in N^q$

consume all of the objects that i^* prefers to b, but not all of the copies of b have been consumed by them. List these objects in the order that their last copies are consumed, and let the set of these objects be $\hat{O} := \{o_1, ..., o_m\} \subset O$, where o_l is completely consumed before o_{l+1} for all $l = 1, \dots, m-1$. (Note that $a \in \hat{O}$.) Let i^{**} be such that $i^* \in \gamma(i^{**})$.

We first construct a correspondence $\xi : \hat{O} \to N^1 \setminus \{i^{**}\}$ defined by

$$\xi(o) := \left\{ i \in N^1 \setminus \{i^{**}\} \mid \exists j \in \gamma(i) \text{ who consumes } o \text{ under } f^q \right\}.$$

Claim 9. Any agent in N^q who consumes o_l prefers o_l to all objects in $\tilde{O} \setminus \{o_1, ..., o_{l-1}\}$ under f^q. Hence, any agent in $\xi(o_l)$ prefers o_l to all objects in $\tilde{O} \setminus \{o_1, ..., o_{l-1}\}$.

Claim 10. For each $O' \subset \hat{O}$, $| \cup_{o \in O'} \xi(o)| \geq |O'|$.

Proof. Suppose otherwise. Then, there exists $O' \subset \hat{O}$ such that $k := |\cup_{o \in O'} \xi(o)| < |O'| =: l$. Reindex the sets so that $\cup_{o \in O'} \xi(o) = \{a^1, ..., a^k\}$ and $O' = \{o^1, ..., o^l\}$. Let x_{ij} denote the number of clones of agent $a^j \in \xi(o^i)$ who consume o^i in the q-fold replica under f^q.

Since $\sum_{i=1}^{l} x_{ij} \leq |\gamma(a^j)| = q$,

$$\sum_{j=1}^{k} \sum_{i=1}^{l} x_{ij} \leq kq.$$

At the same time, all q copies of each object in O' are consumed, and at most $q-1$ clones of i^{**} could be those contributing to that consumption. Therefore,

$$\sum_{i=1}^{l} \sum_{j=1}^{k} x_{ij} \geq lq - (q-1) = (l-1)q + 1 > kq,$$

We thus have a contradiction. ‖

By Hall's Theorem, Claim 10 implies that there exists a mapping $\eta : \hat{O} \to N^1 \setminus \{i^{**}\}$ such that $\eta(o) \in \xi(o)$ for each $o \in \hat{O}$ and $\eta(o) \neq \eta(o')$ for $o \neq o'$.

Now consider the base economy. With positive probability, f^1 has a priority ordering, $(\eta(o_1), ..., \eta(o_m), i^{**})$ followed by an arbitrary permutation of the remaining agents. Given such a priority ordering, the objects in $\hat{O}$ will be all consumed before i^{**} gets her turn but b will not be consumed before i^{**} gets her turn, so she will consume b. This proves that $a \rhd (RP^1, m^1)\, b$. □

Lemma 3. RP^1 is wasteful if and only if RP^q is wasteful for any $q \in \mathbb{N}$.

Proof. We proceed in two steps.

(i) The "only if" Part: Suppose that RP^1 is wasteful. Then, there are objects $a, b \in \tilde{O}$ and an agent $i^* \in N^1$ who prefers a to b such that she consumes b under some ordering $(\tilde{i}^1_{(1)}, \dots, \tilde{i}^1_{(|N^1|)})$ (implied by some $\tilde{f}^1$) and that a is not consumed by any agent under $(\hat{i}^1_{(1)}, \dots, \hat{i}^1_{(|N^1|)})$ (implied by some $\hat{f}^1$). (This is the necessary implication of the "wastefulness" under RP^1.)

Now consider its q-fold replica, RP^q. With positive probability, an ordering $(\bar{\gamma}(\tilde{i}^1_{(1)}), \dots, \bar{\gamma}(\tilde{i}^1_{(|N^1|)}))$ arises, where $\bar{\gamma}(i)$ is an arbitrary permutation of $\gamma(i)$. Clearly, each agent in $\gamma(i^*)$ must consume b even though she prefers a over b (since all copies of all objects the agents in $\gamma(i^*)$ prefer to b are all consumed by the agents ahead of them). Likewise, with positive probability, an ordering $(\bar{\gamma}(\hat{i}^1_{(1)}), \dots, \bar{\gamma}(\hat{i}^1_{(|N^1|)}))$ arises. Clearly, under this ordering, no copies of object a are consumed. It follows that RP^q is wasteful.

(ii) The "if" Part: Suppose next that RP^q is wasteful. Then, there are objects $a, b \in \tilde{O}$ and an agent $i^{**} \in N^q$ who prefers a over b such that she consumes b under some ordering $(\tilde{i}^q_{(1)}, \dots, \tilde{i}^q_{(|N^q|)})$ (implied by some $\tilde{f}^q$) and that not all copies of object a are consumed under $(\hat{i}^q_{(1)}, \dots, \hat{i}^q_{(|N^q|)})$ (implied by some $\hat{f}^q$).

Now consider the corresponding base economy and associated RP^1. The argument of Part (ii) of Lemma 2 implies that there exists an ordering $(\tilde{i}^1_{(1)}, \dots, \tilde{i}^1_{(|N^1|)})$ under which agent $\tilde{i}^* = \gamma^{-1}(i^{**}) \in N^1$ consumes b even though she prefers a over b.

Next, we prove that RP^1 admits a positive-probability ordering under which object a is not consumed. Let $N'' := \{r \in N^1 | \exists j \in \gamma(r) \text{ who consumes the null object under } \hat{f}^q\}$. For each $r \in N''$, we let $\emptyset^r$ denote the null object some clone of $r \in N^1$ consumes. In other words, we use different notations for the null object consumed by the clones of different agents in N''. Given this convention, there can be at most q copies of each $\emptyset^r$.

Let $\bar{O} := O \cup (\cup_{r \in N''} \emptyset^r) \setminus \{a\}$, and define a correspondence $\psi : N^1 \to \bar{O}$ by

$$\psi(r) := \{b \in \bar{O} | \exists j \in \gamma(r) \text{ who consumes } b \text{ under } \hat{f}^q\}.$$

Claim 11. For each $N' \subset N^1$, $| \cup_{r \in N'} \psi(r)| \geq |N'|$.

Proof. Suppose not. Then, $k := | \cup_{r \in N'} \psi(r)| < |N'| =: l$. Reindex the sets so that $\cup_{r \in N'} \psi(r) =: \{o^1, ..., o^k\}$ and $N' = \{r^1, ..., r^l\}$. Let x_{ij} denote the number of copies of object $o^j \in \psi(r^i)$ consumed by the clones of r^i in the q-fold replica under $\hat{f}^q$.

Since there are at most q copies of each object, we must have

$$\sum_{j=1}^{k} \sum_{i=1}^{l} x_{ij} \leq kq.$$

At the same time, all q clones of each agent in N', excluding $q-1$ agents (who may be consuming a), are consuming some objects in O' under $\hat{f}^q$, so we must have

$$\sum_{i=1}^{l} \sum_{j=1}^{k} x_{ij} \geq lq + q - 1 = (l-1)q + 1 > kq,$$

We thus have a contradiction. $\|$

Claim 11 then implies, via Hall's theorem, that there exists a mapping $\iota : N^1 \to \bar{O}$ such that $\iota(r) \in \psi(r)$ for each $r \in N^1$ and $\iota(r) \neq \iota(r')$ if $r \neq r'$.

Let $O' \subset \bar{O}$ be the subset of all object types in $\bar{O}$ whose entire q copies are consumed under $\hat{f}^q$. Order O' in the order that the last copy of each object is consumed; i.e., label $O' = \{o^1, ..., o^m\}$ such that the last copy of object o^i is consumed prior to the last copy of o^j if $i < j$. Let $\hat{N}$ be any permutation of the agents in $\iota^{-1}(\bar{O} \setminus O')$. Now consider the ordering in RP^1: $(\hat{i}^1_{(1)}, \dots, \hat{i}^1_{(|N^1|)}) = (\iota^{-1}(o^1), \dots, \iota^{-1}(o^m), \hat{N})$, where the notational convention is as follows: for any $l \in \{1, \dots, m\}$, if $\iota^{-1}(o^l)$ is empty, then no agent is ordered.

Claim 12. Under the ordering $(\hat{i}^1_{(1)}, \dots, \hat{i}^1_{(|N^1|)}) = (\iota^{-1}(o^1), \dots, \iota^{-1}(o^m), \hat{N})$, a is not consumed.

Proof. For any $l = 0, \dots, m$, let O^l be the set of objects that are consumed by agents $\iota^{-1}(o^1), \dots, \iota^{-1}(o^l)$ under the current ordering (note that some of $\iota^{-1}(o^1), \dots, \iota^{-1}(o^l)$ may be nonexistent). We shall show $O^l \subseteq \{o^1, \dots, o^l\}$ by an inductive argument. First note that the claim is obvious for $l = 0$. Assume that the claim holds for $0, 1, \dots, l-1$. If $\iota^{-1}(o^l) = \emptyset$, then no agent exists to consume an object at this step and hence the claim is obvious. Suppose $\iota^{-1}(o^l) \neq \emptyset$. By definition of ι, agent $\iota^{-1}(o^l)$ weakly prefers o^l to any object in $\tilde{O} \setminus \{o^1, \dots, o^{l-1}\}$. Therefore $\iota^{-1}(o^l)$ consumes an object in $\{o^l\} \cup (\{o^1, \dots, o^{l-1}\} \setminus O^{l-1}) \subseteq \{o^1, \dots, o^l\}$. This and the inductive assumption imply $O^l \subseteq \{o^1, \dots, o^l\}$.

Next, consider agents that appears in the ordered set $\hat{N}$. By an argument similar to the previous paragraph, each agent i in $\hat{N}$ consumes an object in $\iota(i) \cup (\{o^1, \dots, o^m\} \setminus O^m)$. In particular, no agent in $\hat{N}$ consumes a. $\|$

Since the ordering $(\hat{i}^1_{(1)}, \dots, \hat{i}^1_{(|N^1|)}) = (\iota^{-1}(o^1), \dots, \iota^{-1}(o^m), \hat{N})$ realizes with positive probability under RP^1, Claim 12 completes the proof of Lemma 3. □

Proof of Proposition 3. If RP^q is ordinally inefficient for some $q \in \mathbb{N}$, then either it is wasteful or there must be a cycle of binary relation $\triangleright(RP^q, m^q)$. Lemmas 2 and 3 then imply that RP^1 is wasteful or there exists a cycle of $\triangleright(RP^1, m^1)$, and that $RP^{q'}$ is wasteful or there exists a cycle of $\triangleright(RP^{q'}, m^{q'})$ for each $q' \in \mathbb{N}$. Hence, for each $q' \in \mathbb{N}$, $RP^{q'}$ is ordinally inefficient. □

D. Equivalence of Asymmetric RP and PS in Continuum Economies

For $\pi \in \Pi$ and $c \in C$, let $m^\infty_{\pi,c}$ be the measure of agents in class c of preference type π in the continuum economy.

We define asymmetric PS recursively as follows. Let $O^*(0) = \tilde{O}$, $t^*(0) = 0$, and $x_a^*(0) = 0$ for every $a \in \tilde{O}$. Given $O^*(0), t^*(0), \{x_a^*(0)\}_{a\in\tilde{O}}, \ldots, O^*(v-1), t^*(v-1), \{x_a^*(v-1)\}_{a\in\tilde{O}}$, we let $t_\emptyset^* := 1$ and for each $a \in O$, define

$$\text{(D1)} \quad t_a^*(v) = \sup\left\{ t \in [0,1] \,\middle|\, x_a^*(v-1) + \sum_{c\in C} \sum_{\pi: a\in Ch_\pi(O^*(v-1))} \int_{t^*(v-1)}^{t} m_{\pi,c}^\infty g_c(s)ds < 1 \right\},$$

$$\text{(D2)} \quad t^*(v) = \min_{a\in O^*(v-1)} t_a^*(v),$$

$$\text{(D3)} \quad O^*(v) = O^*(v-1) \setminus \{a \in O^*(v-1) | t_a^*(v) = t^*(v)\},$$

$$\text{(D4)} \quad x_a^*(v) = x_a^*(v-1) + \sum_{c\in C} \sum_{\pi: a\in Ch_\pi(O^*(v-1))} \int_{t^*(v-1)}^{t^*(v)} m_{\pi,c}^\infty g_c(t)dt,$$

with the terminal step defined as $\bar{v}^* := \min\{v' | t^*(v') = 1\}$.

Consider the associated expiration dates: For each $a \in \tilde{O}$, $T_a^* := \{t^*(v) | t^*(v) = t_a^*(v), \text{ for some } v\}$ if the set is nonempty, or else $T_a^* := 1$. Let $\tau_a^*(\pi) := \min\{T_a^*, \max\{T_b^* | \pi(b) < \pi(a), b \in O\}\}$ be the expiration date of last object that a type π-agent prefers to a (if it is smaller than T_a^*, and T_a^* otherwise). The **asymmetric PS random assignment in the continuum economy** is defined, for each object $a \in \tilde{O}$, a type-π agent in class c, by $PS_a^*(\pi, c) := \int_{\tau_a^*(\pi)}^{T_a^*} g_c(t)dt$.

In the RP, an agent in class c draws a lottery number $f \in [0,1]$ according to the density function, g_c. Again by the weak law of large numbers, the measure of type-π agents in class c who have drawn lottery numbers less than f is $m_{\pi,c}^\infty \times \int_0^f g_c(f')df'$ (with probability one).

As in the baseline case, the random assignment of RP is described by the cutoff times for the lottery numbers, for alternative objects. And they are described precisely by the same set (D1)-(D4) of equations. In other words, the **random priority random assignment in the continuum economy** is defined, for a type π-agent in class c and $a \in \tilde{O}$, as $RP_a^*(\pi) := T_a^* - \tau_a^*(\pi)$, just as in PS^*. It thus immediately follows that $RP^* = PS^*$, showing that the equivalence extends to the continuum economy with group-specific priorities. The asymptotic equivalence can also be established as explained in the main text, although we omit the proof.

E. Proof of Proposition 4

Let $O = \{a, b\}$, $\Omega = \{\omega_a, \omega_b\}$, $\rho^\infty(\omega_a) = \rho^\infty(\omega_b) = \frac{1}{2}$, agents with π^{ab} prefer a to b to $\emptyset$ and those with π^{ba} prefer b to a to $\emptyset$, $m_{\pi^{ab}}^\infty(\omega_a) = \frac{12}{5}, m_{\pi^{ba}}^\infty(\omega_a) = \frac{8}{5}, m_{\pi^{ab}}^\infty(\omega_b) = \frac{8}{5}, m_{\pi^{ba}}^\infty(\omega_b) = \frac{12}{5}$. Assume for contradiction that mechanism ϕ^* is ordinally efficient and strategy-proof. Since ϕ^* is ordinally efficient, both types of agents prefer both a and b to $\emptyset$, and the measure of all objects (two) is smaller than the measure of all agents (four), at each state the whole measure of both a and b is assigned to agents, that is, $m^\infty_{\pi^{ab}}(\omega)\phi^*_o(\pi^{ab},\omega) + m^\infty_{\pi^{ba}}(\omega)\phi^*_o(\pi^{ba},\omega) = 1$ for every $o \in O$ and $\omega \in \Omega$.

Ordinal efficiency of ϕ^* implies that at most one type of agents receive their non-favorite proper object with positive probability, since otherwise a profitable exchange of probability shares exists either at the same state or across different states. Thus suppose, without loss of generality, that type-π^{ba} agents receive their non-favorite object a with probability zero. Then type-π^{ab} agents obtain the entire share of their favorite object a at both states. Thus,

$$\text{(E1)} \qquad \phi^*_a(\pi^{ab},\omega_a) = \frac{1}{m^\infty_{\pi^{ab}}(\omega_a)} = \frac{5}{12},\ \phi^*_a(\pi^{ab},\omega_b) = \frac{1}{m^\infty_{\pi^{ab}}(\omega_b)} = \frac{5}{8},$$

and

$$\text{(E2)} \qquad \phi^*_a(\pi^{ba},\omega_a) = \phi^*_a(\pi^{ba},\omega_b) = 0.$$

Moreover, since there is mass one of object b,

$$\text{(E3)} \qquad \phi^*_b(\pi^{ba},\omega_a) \le \frac{1}{m^\infty_{\pi^{ba}}(\omega_a)} = \frac{5}{8},\ \phi^*_b(\pi^{ba},\omega_b) \le \frac{1}{m^\infty_{\pi^{ba}}(\omega_b)} = \frac{5}{12}.$$

If a type π^{ba}-agent reports true preferences π^{ba}, then by (E2) and (E3),

$$\Phi^*_a(\pi^{ba}) + \Phi^*_b(\pi^{ba}) = 0 + \bar{P}(\omega_a|\pi^{ba})\phi^*_b(\pi^{ba},\omega_a) + \bar{P}(\omega_b|\pi^{ba})\phi^*_b(\pi^{ba},\omega_b) \le \frac{1}{2},$$

where $\bar{P}(\omega|\pi)$ denotes the posterior belief of an agent that the state is ω given that her preference type is π.

On the other hand, if she lies and reports π^{ab}, then by (E1) she expects to obtain object a with probability

$$\bar{P}(\omega_a|\pi^{ba})\phi^*_a(\pi^{ab},\omega_a)+\bar{P}(\omega_b|\pi^{ba})\phi^*_a(\pi^{ab},\omega_b) = \frac{4}{10}\cdot\frac{5}{12}+\frac{6}{10}\cdot\frac{5}{8} = \frac{13}{24} > \frac{1}{2} \ge \Phi^*_a(\pi^{ba})+\Phi^*_b(\pi^{ba}),$$

violating strategy-proofness of ϕ^*.

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