Lorenz rankings of rules for the adjudication of conflicting claims

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Abstract

For the problem of adjudicating conflicting claims, we offer simple criteria to compare rules on the basis of the Lorenz order. These criteria pertain to three families of rules. The first family contains the constrained equal awards, constrained equal losses, Talmud, and minimal overlap rules (Thomson, 2007a). The second family, which also contains the constrained equal awards and constrained equal losses rules, is obtained from the first one by exchanging, for each problem, how well agents with relatively larger claims are treated as compared to agents with relatively smaller claims (Thomson, 2007a). The third family comprises all consistent rules (Young, 1987). We also address the issue whether certain operators on the space of rules (Thomson and Yeh, 2001)preserve the Lorenz order, or reverse it.

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1 Introduction

When several agents have claims on a resource adding up to more than is available, what should be done with what is available of the resource? For example, when a firm goes bankrupt, how should its liquidation value be divided among its creditors? A "division rule" is a function that associates with each such "claims problem" a recommendation for it. Alternatively, think of a tax authority having to raise a certain amount of money from taxpayers whose incomes differ, in order to cover the cost of some public project. How much should each taxpayer be assessed? This problem is mathematically identical to the problem of claims adjudication, but we will use language that fits that first application, calling the data defining a problem "claims" and "endowment", and the amount assigned to an agent his "award".[1]

An important issue when evaluating a rule is how differentially it treats relatively larger claimants as compared to relatively smaller claimants. The Lorenz order is commonly used to evaluate income distributions and it is just as natural in the context of claims adjudication. Thus, our goal is to develop general criteria to perform Lorenz comparisons of rules. Because rules are complex objects however, one should not expect to be able to make Lorenz comparisons of any two rules very generally, independently of the data of the problem. The comparisons may depend on the specific values claims take, the endowment, the number of claimants, and even their identities. In fact, for each particular problem, the awards vectors chosen by two rules may not be Lorenz comparable. Nevertheless, one may hope that within structured classes of rules, such comparisons would be feasible. Our main results fulfill this hope. Indeed, they concern three families that are sufficiently wide as to contain all of the rules that have been most often discussed in the literature. Some also exploit the fact that the space of rules can be structured by "operators".

Although a great variety of rules have been proposed, motivated by a broad range of considerations, several unifying principles can be invoked to organize them into families. We start with a family (Thomson, 2000) whose definition is quite simple and intuitive—and in fact it expresses a fairly widely held view of how this type of problems should be solved—yet it is rich enough to contain a number of important rules. They are the "constrained equal awards rule" and the "constrained equal losses rule", both already

[1]For surveys of the literature on this subject, see Thomson (2003, 2007c).

familiar to Maimonides (12th Century); the "Talmud rule" (Aumann and Maschler, 1985), proposed to rationalize certain numerical examples found in the Talmud (O'Neill, 1982); and the "minimal overlap rule" (O'Neill, 1982), which is an extension[2] of an incompletely specified medieval rule (Rabad, 12th Century), and which at first sight, does not appear to be related to the other three. The second family is a counterpart of the first family in which relatively small claims and relatively large claims are treated in a reverse way to how they are treated by members of the first family. It contains the constrained equal awards and constrained equal losses rules. (These two rules are the only ones the two families have in common.) The main ingredient in the definitions of the two families is the basic idea of equality, applied recursively either to the amounts claimants receive or to the losses they incur.

The third family comprises all the rules satisfying the property called "consistency".[3] Let a rule be given. Consider a problem, and apply the rule to solve it. Then imagine that some claimants leave with their awards, and reassess the situation. The requirement is that for the resulting "reduced" problem—in that problem, the endowment is what it was initially minus what the agents who left took with them and the claims of the remaining agents are what they were initially—the rule should assign to each claimant the same amount as it did initially. Several of the aforementioned examples, such as the constrained equal awards, constrained equal losses, and Talmud rules are consistent. So are Piniles' rule (Piniles, 1861), which was an early, only partially successful, attempt at generating the numbers in the Talmud, the constrained egalitarian rule (Chun, Schummer, and Thomson, 2001), which offers another implementation of the idea of equality subject to constraints, and of course the proportional rule. (The minimal overlap rule is not however.)

For each of the three families, we formulate a general and simple criterion that allows Lorenz comparisons of their members. A special case of our result concerning the first family is that the Talmud rule Lorenz dominates the minimal overlap rule. Another is a complete Lorenz ranking of the members of a one-parameter family of rules, a subfamily of the ICI family, proposed by Moreno-Ternero and Villar (2006b) as a generalization of the constrained

[2]Another extension of Ibn Ezra's suggestion is proposed by Bergantiños and Méndez-Naya (2001) and Alcalde, Marco, and Silva (2005).

[3]For a survey of the applications of the "consistency principle", see Thomson (2007b).

equal awards, constrained equal losses, and Talmud rules (Section 4). Such a ranking is obtained by Moreno-Ternero and Villar (2006a) by direct calculation. We also derive a complete Lorenz ranking of the members of a one-parameter subfamily of the second family defined in parallel manner. (In both cases, the parameter is a point in the unit interval.)

Our results pertaining to the family of consistent rules involve proving that, adapting Hokari and Thomson's (2000) terminology, Lorenz domination is "lifted" by consistency from the two-claimant case to the case of arbitrarily many claimants for all rules satisfying a pair of elementary properties: lifting of an order means that given two consistent rules, if one of them dominates (in that order) the other for two claimants, its domination extends to arbitrarily many claimants. Then, to establish Lorenz domination in general, it suffices to check the two-claimant case, which is trivial. Applications of this result are numerous because many rules are consistent, as we noted. As corollaries, we obtain rankings of several important ones. We also recover the Lorenz ranking of the members of the Moreno-Ternero–Villar family (2006b) and the ranking of the reverse family (Section 5).

We complete this study by exploiting the notion of an "operator" on the space of rules. This is a mapping that associates with each rule another one (Thomson and Yeh, 2001). Here is not the place to go into the details but let us simply say that operators provide another very useful way of structuring the space of rules, in relating rules and properties of rules, and in generating theorems. The examples below should make it clear how they do so. Now, given two rules that are Lorenz ordered, one should know if their images under an operator are ordered too. We consider four operators. Given a rule S, the "claims truncation operator" associates with S the rule obtained, for each problem, by applying S to the derived problem in which claims have been truncated at the endowment. The "attribution of minimal rights operator" associates with S the rule that, for each problem, calculates the awards vector in two steps: first, each agent receives the difference between the endowment and the sum of the claims of the others (or 0 if this difference is negative); second, he receives what S would give him in the division of what remains of the endowment, claims having been adjusted down by the first installments. The "duality operator" associates with S the rule that, for each problem, divides the endowment as S divides the deficit (namely the difference between the sum of the claims and the endowment). We show that the first operator preserves the Lorenz order and that the second one generally preserves it.

It is already known that the third one generally reverses it (Hougaard and Thorlund-Petersen, 2001). We also provide a Lorenz domination result pertaining to each rule satisfying some basic properties and the rule obtained by subjecting it to any one of these operators. Finally, we consider the family of operators that associate with each list of rules and each list of non-negative weights for them, the weighted average of these rules. These operators also generally preserve Lorenz domination (Section 6).

Bosmans and Lauwers (2007)'s objective is also to obtain Lorenz rankings of rules, but the main thrust of their approach is to identify a certain rule as being Lorenz minimal (or Lorenz maximal) within a certain class of rules. The class is defined by means of properties its members satisfy. They offer several characterizations of this type. Some of the rankings they obtain as corollaries are also derived in the present paper. We comment in greater detail on their contribution at the relevant points. Of course, some rules cannot be Lorenz ordered. Showing that, given a pair of rules, neither one Lorenz dominates the other is achieved by means of examples of problems. A number of such examples can be found in their paper, settling the issue of dominance in the negative for several important pairs.

There has been a growing interest in the distributional properties of division rules. Hougaard and Østerdal (2005) and Kasajima and Velez (2008) ask whether a possible Lorenz ranking of claims vectors is reflected, for each endowment, in a Lorenz ranking of the awards vectors chosen for the resulting problems. The requirement that this be so is first suggested by Hougaard and Thorlund-Petersen (2001). Ju and Moreno-Ternero (2006, 2007) are concerned with the derivation of conditions under which an awards vector chosen for a problem can be seen as more equal than the claims vector of that problem. Thus, these various authors address questions that can be seen as complementary to the ones we are asking here.

The Lorenz criterion is certainly not the only concept that can help in illuminating the distributional properties of rules. In the context of taxation, one speaks of progressive rules and of regressive rules, of their degree of progressivity or regressivity, and a large literature has also been devoted to the derivation of indices to measure income inequality and understanding how it is affected by taxation. We are in the privileged position to be able to draw on the conceptual apparatus developed in that field. In the current context, the shape of paths of awards has also be used to get a sense of who a rule favors. The path can be convex, concave, "visible" from below, "visible" from above, and these properties can be met weakly or strictly. In the previous paragraph, we mentioned an emerging literature devoted to understanding the problem and our paper should be seen as a contribution to this effort. Much remains to be done. In particular, there has been so far no application of the indices of income inequality developed in public finance.

Second, we would like to emphasize that we do not view as the theorist's mandate to take a position on how progressive taxation should be, nor, when two rules can be ranked on the basis of their progressivity or their regressivity, or by means of the Lorenz criterion, to judge which is preferable. It is for the political process to sort these issues out. Some of the results that have been obtained in the context of claims adjudication are characterizations of particular rules as being Lorenz "maximal", or Lorenz "minimal" within certain classes of rules. These results should not been interpreted as revealing the superiority or inferiority of these rules within these classes. By offering definitions, uncovering how they are related, and working out their implications, the public finance theorist helps the tax authority and the voting public formalize intuition about how rules treat large incomes in relation to small incomes. Our objective is the same. It appears to be harder to reach because in the context of claims adjudication, a much greater variety of rules have been found worth studying and ranking them is a more daunting task. Nevertheless, as we have shown, a large number of Lorenz rankings can indeed be obtained.

Finally, and although our study cannot by any means be considered as axiomatic, we would like to point out how useful previous axiomatic work has been to us. This is because this work has contributed importantly to structuring the space of rules, and it is by exploiting these structures that we have been able to make progress. The most striking illustration is our appeal to the variable population concept of consistency. Even though our results apply to a fixed population of agents, we have obtained some of them by embedding the problem into a framework in which the population of claimants may vary. We could thereby invoke this property, which has to do with how the recommendations rules make vary as population varies, and deduce in a remarkably easy way Lorenz rankings for the general case from Lorenz rankings in the trivial two-agent case.

2 The model of claims resolution. Lorenz domination

A group of agents, $N = \{1, \ldots, n\}$ with $n \geq 2$, have claims on a resource, $c_i \in \mathbb{R}_+$ being the **claim** of agent $i \in N$ and $c \equiv (c_i)_{i \in N}$ the claims vector. Claims are ordered so that $c_1 \leq \cdots \leq c_n$. There is an **endowment** $\boldsymbol{E}$ of the resource that is insufficient to honor all of the claims. Using the notation $\mathbb{R}^N_+$ for the cross-product of n copies of $\mathbb{R}_+$ indexed by the members of N, a claims problem, or simply a **problem**, is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum c_i \geq E$.[4] Let $\boldsymbol{\mathcal{C}^N}$ denote the domain of all problems.

A division rule, or simply a **rule**, is a function that associates with each problem $(c, E) \in \mathcal{C}^N$ a vector $x \in \mathbb{R}^N$ satisfying the non-negativity and claims boundedness inequalities $0 \leqq x \leqq c$ and the efficiency equality $\sum x_i = E$.[5] Such an x is an **awards vector for** $\boldsymbol{(c, E)}$. Let $\boldsymbol{X(c, E)}$ be the set of these vectors.

An important way to compare rules is by means of how evenly distributed the awards they select are. Is a rule relatively favorable to large claimants or to small claimants? In the theory concerning the measurement of income inequality, a number of partial orders have been defined to perform comparisons of income distributions.[6] The central one is based on the successive sums of ordered incomes, and the obvious way to apply it here is to consider successive sums of ordered awards, as follows. Let x and $y \in \mathbb{R}^N$ be such that $x_1 \leq \cdots \leq x_n$, $y_1 \leq \cdots \leq y_n$, and $\sum x_i = \sum y_i$. Then, $\boldsymbol{x}$ **is greater than** $\boldsymbol{y}$ **in the Lorenz order**, which we write as $x \succ_L y$, if $x_1 \geq y_1$, $x_1 + x_2 \geq y_1 + y_2$, $x_1 + x_2 + x_3 \geq y_1 + y_2 + y_3$, and so on, with at least one strict inequality. We also say then that $\boldsymbol{x}$ **Lorenz dominates** $\boldsymbol{y}$. If the successive partial sums are equal for the two vectors, x and y are **Lorenz equivalent**. If either $x \succ_L y$ or these vectors are Lorenz equivalent, we write $x \succeq_L y$. **A rule** $\boldsymbol{S}$ **Lorenz dominates a rule** $\boldsymbol{S'}$ if for each $(c, E) \in \mathcal{C}^N$, $S(c, E) \succeq_L S'(c, E)$.

[4] A summation without explicit bounds should be understand to be carried out over all claimants.

[5] Vector inequalities: $x \geqq y$, $x \geq y$, and $x > y$.

[6] One is based on the gap between the smallest and largest awards; another on the variance of the awards. Comparisons of rules on the basis of either gap or variance are presented by Schummer and Thomson (1997) and Chun, Schummer, and Thomson (2001).

3 Rules and families of rules

First, we describe several rules, and families of rules, that will be covered by our main theorems.

3.1 Four central rules

The following rules are central in the literature.

Constrained equal awards rule, *CEA*: For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, $CEA_i(c, E) \equiv \min\{c_i, \lambda\}$, where $\lambda \in \mathbb{R}_+$ is chosen so as to achieve efficiency.

Constrained equal losses rule, *CEL*: For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, $CEL_i(c, E) \equiv \max\{0, c_i - \lambda\}$, where $\lambda \in \mathbb{R}_+$ is chosen so as to achieve efficiency.

Talmud rule, *T*: (Aumann and Maschler, 1985) For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$,

$$T_i(c, E) \equiv \begin{cases} \min\{\frac{c_i}{2}, \lambda\} & \text{if } \sum \frac{c_j}{2} \geq E, \\ c_i - \min\{\frac{c_i}{2}, \lambda\} & \text{otherwise,} \end{cases}$$

where in each case, $\lambda \in \mathbb{R}_+$ is chosen so as to achieve efficiency.

Minimal overlap rule, *MO*: (O'Neill, 1982) Claims on specific parts of the endowment are arranged so that the part claimed by exactly one claimant (whoever that claimant is; different "subparts" may be claimed by different claimants) is maximized, and for each $k = 2, \ldots, n-1$ successively, subject to the first k maximizations being solved, the part claimed by exactly $k + 1$ claimants (whoever these claimants are; different subparts may be claimed by different groups of $k + 1$ claimants) is maximized. Once claims are arranged in this way, for each part of the endowment, equal division prevails among all agents claiming it. Each claimant receives the sum of the partial compensations assigned to him for the various parts that he claimed.

Formulas for the rule are available (O'Neill, 1982; Chun and Thomson, 2005; Alcalde, Marco, and Silva, 2007), as well as a characterization based on an additivity condition (Marchant, 2008).

3.2 The ICI family

The considerations underlying the definition of the minimal overlap rule seem far removed from those that underlie the constrained equal awards, constrained equal losses, and Talmud rules. However, by plotting for each claims vector the awards they recommend as a function of the endowment, one notes similarities. The award to each claimant increases initially, then it remains constant for a while, then it increases again until he is fully compensated; given two claimants, the interval of constancy for the smaller claimant contains the interval of constancy for the larger claimant; finally, any two awards that are increasing at any given moment do so at the same rate. Let us consider all rules exhibiting these features. We designate the family they constitute by the name of "Increasing-Constant-Increasing" family, or **ICI family** for short (the interval in which each claimant's award is increasing can be subdivided into subintervals in which the rate of increase is constant; for the largest claimant, the interval in which his award stays fixed is degenerate).

Proceeding in this way is easily justified by recursive application of the widely held view that for a small endowment, differences in claims should be judged irrelevant. Then equal shares are appealing as providing "consolation prizes". Starting with equal division, as E increases, at some point—let $F_1(c)$ denote the value of E at which this occurs—claimant 1's award is perceived as being too large in relation to his claim, so he is dropped off. Differences in the claims of the others might still be judged irrelevant for a while, and we continue with equal division for them until—let $F_2(c)$ denote the value of E at which this occurs—it is felt that claimant 2's award is too large in relation to his claim. Then, he is dropped off too and we continue with equal division for the remaining $n-2$ claimants, and so on. We proceed until claimant $n-1$ has been dropped off—let $F_{n-1}(c)$ denote the value of E at which this occurs. To complete the description of the rule, it is convenient to start with $E = \sum c_i$ and let E decrease (and the deficit, namely the difference $\sum c_i - E$, grows). This time, we focus on the losses claimants incur. The deficit is initially divided equally among all claimants until it is felt that claimant 1's loss is too large in relation to his claim—let $G_1(c)$ denote the value of E at which this occurs. The next increments in the deficit are divided equally among claimants $2, \ldots, n$ until it is felt that claimant 2's loss is too large in relation to his claim and he is dropped off too—let $G_2(c)$ denote the value of E at which this occurs; we proceed until claimant $n-1$ is dropped off—let $G_{n-1}(c)$ denote the value of E at which this occurs. When E decreases from $G_{n-1}(c)$ to $F_{n-1}(c)$, claimant n is the only one to absorb incremental deficits. Equivalently, when E increases from $F_{n-1}(c)$ to $G_{n-1}(c)$, claimant n is the only recipient of each new unit.

The freedom in defining the rules comes from that of choosing the various points at which claimants are dropped off and picked up again, and the fact that these drop-off and pick-up points may depend on the claims vector. These parameters are not independent however. Indeed, claimant 1's award increases twice, at Step 1 by $\frac{F_1(c)}{n}$, and at Step $2n-1$ (the last step) by $\frac{\sum c_i - G_1(c)}{n}$, for a total of c_1. Thus, $\frac{F_1(c)}{n} + \frac{\sum c_i - G_1(c)}{n} = c_1$. Claimant 2's award increases along with claimant 1's award on both occasions and at the same rate, also for a total of c_1; in addition, it increases at Step 2 by $\frac{F_2(c)-F_1(c)}{n-1}$ and at Step $2n-2$ (the penultimate step) by $\frac{G_1(c)-G_2(c)}{n-1}$. Thus, $c_1 + \frac{F_2(c)-F_1(c)}{n-1} + \frac{G_1(c)-G_2(c)}{n-1} = c_2$. Similar statements can be made about the other claimants. For the last claimant, we obtain $c_{n-1} + G_{n-1}(c) - F_{n-1}(c) = c_n$.

Summarizing, here is the general definition of the family. Its members are indexed by a list $H \equiv (F_k, G_k)_{k=1}^{n-1}$ of pairs of functions from $\mathbb{R}^N_+$ to $\mathbb{R}_+$ such that for each $c \in \mathbb{R}^N_+$, $(F_k(c))_{k=1}^{n-1}$ is nowhere decreasing, $(G_k(c))_{k=1}^{n-1}$ is nowhere increasing, $G_1(c) \leq \sum c_i$, and the following **ICI relations** hold. For convenience, we introduce $c_0 \equiv 0$, $F_0(c) \equiv 0$, $G_0(c) \equiv \sum c_i$, and $F_n(c) = G_n(c)$. Then, for each $k = 1, \ldots, n$,

$$c_{k-1} \quad + \quad \frac{F_k(c)-F_{k-1}(c)}{n-k+1} \quad + \quad \frac{G_{k-1}(c)-G_k(c)}{n-k+1} \quad = \quad c_k.$$

The ICI relations are linearly dependent; multiplying the first one through by n, the second one by $n-1$, ..., and the last one by 1, gives new relations whose sum is an identity. If several agents have equal claims, the ICI relations imply that successive $F_k(c)$'s are equal and that so are the corresponding $G_k(c)$'s. Then, agents with equal claims drop out together and they come back together. Let $\boldsymbol{\mathcal{H}^N}$ denote the family of lists $H \equiv (F_k, G_k)_{k=1}^{n-1}$ of pairs of functions satisfying the ICI relations. Here is the formal definition of the ICI family (Thomson, 2007a):

ICI rule relative to $\boldsymbol{H \equiv (F_k, G_k)_{k=1}^{n-1} \in \mathcal{H}^N}$, $\boldsymbol{S^H}$: For each $c \in \mathbb{R}^N_+$ with $c_1 \leq \cdots \leq c_n$, the awards vector chosen by S^H is calculated as follows. As E first increases from 0 to $F_1(c)$, equal division prevails; as E increases from $F_1(c)$ to $F_2(c)$, claimant 1's award remains constant, and each increment is divided equally among the others. As E increases from $F_2(c)$ to $F_3(c)$, claimants 1 and 2's awards remain constant, and each increment is divided equally among the others... This goes on until E reaches $F_{n-1}(c)$. As E increases from $F_{n-1}(c)$ to $G_{n-1}(c)$, each increment goes entirely to claimant n. As E increases from $G_{n-1}(c)$ to $G_{n-2}(c)$, each increment is divided equally between claimants n and $n-1$... This goes on until E reaches $G_1(c)$, after which each increment is divided equally among all claimants, until all are fully compensated.

The four rules defined of Subsection 3.1 are ICI rules and our first lemma identifies functions in $\mathcal{H}^N$ that rationalize them as members of the family.[7]

Lemma 1 (Thomson, 2000) *The following four rules are ICI rules:*

(a) CEA: for each $c \in \mathbb{R}^N_+$, *set* $F(c) \equiv (nc_1, c_1 + (n-1)c_2, \ldots, c_1 + c_2 + \cdots + c_{k-1} + (n-k+1)c_k, \ldots, c_1 + c_2 + \cdots + c_{n-2} + 2c_{n-1})$.

(b) CEL: for each $c \in \mathbb{R}^N_+$, *set* $F(c) \equiv (0, \ldots, 0)$.

(c) T: for each $c \in \mathbb{R}^N_+$, *set* $F(c) \equiv (n\frac{c_1}{2}, \frac{c_1}{2} + (n-1)\frac{c_2}{2}, \ldots, \frac{c_1}{2} + \frac{c_2}{2} + \cdots + \frac{c_{k-1}}{2} + (n-k+1)\frac{c_k}{2}, \ldots, \frac{c_1}{2} + \frac{c_2}{2} + \cdots + \frac{c_{n-2}}{2} + 2\frac{c_{n-1}}{2})$.

(d) MO: for each $c \in \mathbb{R}^N_+$, *set* $F(c) = (c_1, c_2, \ldots, c_{n-1})$.

Moreno-Ternero and Villar (2006a,b) propose and study a one-parameter family of rules under the name of "Tal family". This family happens to be a subfamily of the ICI family defined by selecting $\theta \in [0, 1]$ and choosing $F(c) \equiv \theta(nc_1, c_1 + (n-1)c_2, \ldots, c_1 + c_2 + \cdots + c_{k-1} + (n-k+1)c_k, \ldots, c_1 + c_2 + \cdots + c_{n-2} + 2c_{n-1})$. Let $\boldsymbol{T^\theta}$ denote the rule associated with θ in this manner. Note that $T^1 = CEA$, $T^0 = CEL$, and $T^{\frac{1}{2}} = T$.

3.3 Reverse family: the CIC family

A reverse algorithm to the one underlying the definition of the ICI family suggests itself. Fix the claims vector. There, we began with equal division and dropped claimants in succession, starting with the smallest claimant. Here, as the endowment increases from 0 to the sum of the claims, we start by giving everything to claimant n and progressively enlarge the set of recipients, introducing them in the order of decreasing claims. More precisely, claimant n is the only one present until E reaches a first critical value—let $F_1(c)$ denote this value—at which point claimant $n-1$ enters the scene. Then, each increment is divided equally between claimants n and $n-1$ until

[7]The alternative extension of Rabad's rule mentioned in footnote 2 is not an ICI rule.

E reaches a second critical value—let $F_2(c)$ denote this value. Then, each increment is divided equally among claimants n, $n-1$, and $n-2$, and so on. This process goes on until claimant 1 enters the scene—let $F_{n-1}(c)$ denote the value of E at which this occurs—at which point each increment is divided equally among all claimants. At some point—let $G_{n-1}(c)$ denote the value of E at which this occurs—claimant 1 is fully compensated and is dropped off. Then, each increment is divided equally among claimants 2 through n until claimant 2 is fully compensated—let $G_{n-2}(c)$ denote the value of E at which this occurs—and is dropped off and so on. During the last step, claimant n is the only one left.

A member of the family defined in this way can be described in terms of a list of pairs of functions satisfying relations parallel to those imposed on the pairs defining the ICI rules. Specifically, consider a list $H \equiv (F_k, G_k)_{k=1}^{n-1}$ of pairs of functions from $\mathbb{R}_+^N$ to $\mathbb{R}_+$ such that for each $c \in \mathbb{R}_+^N$, $(F_k(c))_{k=1}^{n-1}$ is nowhere decreasing, $(G_k(c))_{k=1}^{n-1}$ is nowhere increasing, $G_1(c) \leq \sum c_i$, and the following relations, which we call the **CIC relations**, hold. The acronym CIC reflects the fact that each claimant's award is first Constant, then Increasing, then Constant (except for the largest claimant, as his award is always increasing). For convenience, we introduce $c_0 \equiv 0$, $F_0(c) \equiv 0$, $G_0(c) \equiv \sum c_i$, $F_n(c) = G_n(c)$. Then, for each $k = 1, \ldots, n$,

$$c_{k-1} \quad + \quad \frac{F_{n-k+1}(c) - F_{n-k}(c)}{n-k+1} \quad + \quad \frac{G_{n-k}(c) - G_{n-k+1}(c)}{n-k+1} \quad = \quad c_k.$$

The CIC relations are imposed to guarantee that as the endowment reaches the sum of the claims, each claimant is fully compensated. As was the case for the ICI relations, the CIC relations are not independent: multiplying the first one through by n, the second one by $n-1$, ..., and the last one by 1, gives new relations whose sum is an identity. Also, if several agents have equal claims, the CIC relations imply that successive F_k's are equal and that so are the corresponding G_k's. This means that as the endowment increases, agents with equal claims come in together and they drop out together.

Let $\bar{\mathcal{H}}^N$ denote the family of lists of pairs of functions satisfying the CIC relations. Here is the formal definition of the CIC family (Thomson, 2007a):

CIC rule relative to $H \equiv (F_k, G_k)_{k=1}^{n-1} \in \bar{\mathcal{H}}^N$, R^H: For each $c \in \mathbb{R}_+^N$ with $c_1 \leq \cdots \leq c_n$, the awards vector chosen by R^H is calculated as follows. As E first increases from 0 to $F_1(c)$, everything goes to claimant n. As

E increases from $F_1(c)$ to $F_2(c)$, equal division of each increment prevails between claimants n and $n-1$. As E increases from $F_2(c)$ to $F_3(c)$, equal division of each increment prevails among claimants n, $n-1$, and $n-2$... This goes on until claimant 1 enters the scene, when $E = F_{n-1}(c)$, at which point equal division of each increment prevails until, when $E = G_{n-1}(c)$, he is fully compensated and is dropped off. As E increases from $G_{n-1}(c)$ to $G_{n-2}(c)$, equal division of each increment prevails among claimants 2 through n. Then, claimant 2 is fully compensated and is dropped off... At the end of the process, claimant n is the only one left, and he receives each increment until he is fully compensated.

The next lemma identifies three rules as CIC rules. The constrained equal awards and constrained equal losses rules are two of them. The third one is defined like the Talmud rule, which can be seen as a "hybrid" of the constrained equal awards and constrained equal losses rules, by exchanging the order in which these component rules are applied, with the half-claims being used in the formulas instead of the claims themselves. This reversal is the reason for the name we chose for it.[8]

Reverse Talmud rule, $\boldsymbol{T^r}$: For each $(c, E) \in \mathcal{C}^N$,

$$T_i^r(c,E) \equiv \begin{cases} \max\{\frac{c_i}{2} - \lambda, 0\} & \text{if } E \leq \sum \frac{c_j}{2}, \\ \frac{c_i}{2} + \min\{\frac{c_i}{2}, \lambda\} & \text{otherwise,} \end{cases}$$

where in each case, $\lambda \in \mathbb{R}_+$ is chosen so as to achieve efficiency.

Lemma 2 (Thomson, 2000) *The following rules are CIC rules:*

(a) CEA: for each $c \in \mathbb{R}_+^N$, set $F(c) \equiv (0, 0, \ldots, 0)$.

(b) CEL: for each $c \in \mathbb{R}_+^N$, set $F(c) \equiv (c_n - c_{n-1}, c_n + c_{n-1} - 2c_{n-2}, \ldots, c_n + c_{n-1} + \cdots + c_2 - (n-1)c_1)$.

(c) T^r: for each $c \in \mathbb{R}_+^N$, set $F(c) \equiv (-\frac{c_{n-1}}{2} + \frac{c_n}{2}, -2\frac{c_{n-2}}{2} + \frac{c_{n-1}}{2} + \frac{c_n}{2}, \cdots, -k\frac{c_{n-k}}{2} + \frac{c_{n-k+1}}{2} + \cdots + \frac{c_n}{2}, \cdots, -(n-1)\frac{c_1}{2} + \frac{c_2}{2} + \cdots + \frac{c_n}{2})$.

Generalizing the reverse Talmud rule, a counterpart of family $\{T^\theta\}_{\theta \in [0,1]}$ can be defined by reversing the order in which the ideas of equal gains and equal losses are applied. For each $c \in \mathbb{R}_+^N$, set $F(c) \equiv \theta(c_n - c_{n-1}, c_n + c_{n-1} - 2c_{n-2}, \ldots, c_n + c_{n-1} + \cdots + c_2 - (n-1)c_1)$. Let $\boldsymbol{\{U^\theta\}_{\theta \in [0,1]}}$ denote this family. Note that $U^0 = CEA$, $U^1 = CEL$, and $U^{\frac{1}{2}} = T^r$.

[8] It is discussed under that name by Chun, Schummer, and Thomson (2001) and Hokari and Thomson (2003).

4 Lorenz comparisons of ICI rules and Lorenz comparisons of CIC rules

Our first theorem formulates a simple criterion on pairs of ICI rules that tells us when they can be Lorenz ordered. It suffices to check that the F-sequence of one dominates the F-sequence of the other; equivalently, that the G-sequence of one dominates the G-sequence of the other. Its proof, as well as the proofs of most of the forthcoming results, are in the appendix.

Theorem 1 *Let S and S' be two ICI rules, associated with $H \equiv (F, G)$ and $H' \equiv (F', G') \in \mathcal{H}^N$ respectively. If, for each $k \in \{1, \dots, n-1\}$, $F'_k \geq F_k$, then S' Lorenz dominates S. Equivalently, it suffices that for each $k \in \{1, \dots, n-1\}$, $G'_k \geq G_k$.*

Lorenz rankings of several rules are obtained as easy corollaries of Theorem 1:

Corollary 1 *(a) The constrained equal awards rule Lorenz dominates all ICI rules.*

(b) All ICI rules Lorenz dominate the constrained equal losses rule.

(c) The Talmud rule Lorenz dominates the minimal overlap rule.

(d) For each pair $\{\theta^1, \theta^2\}$ of elements of $[0, 1]$, if $\theta^1 \leq \theta^2$, then T^{θ^2} Lorenz dominates T^{θ^1}.

Proof: Given an ICI rule S, let $H^S \equiv (F^S, G^S) \in \mathcal{H}^N$ denote the function associated with it. We show that in each case, and for each $c \in \mathbb{R}^N_+$, the inequalities of Theorem 1 are met.

(a) Here, it is simpler to compare the $G_k(c)$'s. For each ICI rule S, we have $G^S(c) \leq G^{CEA}(c) = (\sum c_i, \dots, \sum c_i)$.

(b) For each ICI rule S, we have $F^S(c) \geq F^{CEL}(c) = (0, \dots, 0)$.

(c) The sequences $(F^T_k(c))_{k=1}^{n-1}$ and $(F^{MO}_k(c))_{k=1}^{n-1}$ are given in Lemma 1. We compare them term by term:

$$\begin{array}{lllllllll}
n\frac{c_1}{2} & & & & & & & \geq & c_1 \\
\frac{c_1}{2} & + & (n-1)\frac{c_2}{2} & & & & & \geq & c_2 \\
\cdots & & & & & & & \geq & \cdots \\
\frac{c_1}{2} & + & \frac{c_2}{2} & + \cdots + & (n-k+1)\frac{c_k}{2} & & & \geq & c_k \\
\cdots & & & & & & & \geq & \cdots \\
\frac{c_1}{2} & + & \frac{c_2}{2} & + \cdots + & \cdots & + & 2\frac{c_{n-1}}{2} & \geq & c_{n-1}
\end{array}$$

The first inequality is obviously satisfied. The remaining ones can be written, for each $k = 2, \ldots, n-1$, as $\sum_{\ell=1}^{k-1} c_\ell + (n-k-1)c_k \geq 0$.

(d) The functions $F^{T^{\theta^1}}$ and $F^{T^{\theta^2}}$ are non-negative and proportional. Thus, they trivially satisfy the hypotheses of Theorem 1. □

We need to introduce additional properties of rules to relate Corollary 1 to existing literature. **Order preservation of awards** (Aumann and Maschler, 1985) says that for each problem, given any two claimants, the larger claimant should be awarded at least as much as the smaller claimant, and **order preservation of losses** (Aumann and Maschler, 1985) says that under the same hypotheses, the former should incur a loss that is at least as large as the loss incurred by the latter. **Order preservation** is the conjunction of the two requirements.

Parts (a) and (b) of Corollary 1 are not the interesting ones, as more general results can easily be proved: first, the constrained equal awards rule Lorenz dominates all rules. (Thorlund-Peterson, 2001; Moreno-Ternero and Villar, 2006b, state domination among rules satisfying *order preservation*, as this property is incorporated in their definition of a rule; Bosmans and Lauwers, 2007, state this domination result.)

Also, the constrained equal losses rule is Lorenz-dominated by all rules satisfying *order preservation* (Bosmans and Lauwers, 2007). [9] Bosmans and Lauwers (2007) obtain (c) as a corollary of a result stating that the minimal overlap rule is Lorenz minimal in a class of rules satisfying a certain list of properties, all of which are satisfied by the Talmud rule.[10] A direct proof of Part (d) is given by Moreno-Ternero and Villar (2006b).

[9] To show that the full strength of *order preservation* is needed), let $N \equiv \{1,2\}$ and consider a rule whose path for $c \equiv (1,2)$ is seg$[(0,0),(0,2),c]$. This rule, which does not Lorenz dominate the constrained equal losses rule, violates *order preservation of losses*. Also, consider a rule whose path for $c \equiv (2,3)$ is seg$[(0,0),(2,0),c]$. This rule, which does not Lorenz dominate the constrained equal losses rule (to see this, set $E = 2$ for instance), violates *order preservation of awards*.

[10] Consider the family of rules satisfying the *reasonable lower bound* (which says that each agent's award should be at least as large as $1/n$ times his claim truncated at the endowment (Moreno-Ternero and Villar, 2004; Dominguez and Thomson, 2006), *order preservation*, *order preservation under claims variations* (which says that if some agent's claim increases, given two other agents, the one with the larger claim should incur a loss that is at least as large as the loss incurred by the one with the smaller claim; Thomson, 2007b), and *limited consistency* (which says that if an agent's claim is 0 and he leaves, the awards to the other claimants should not be affected; Section 5 is devoted to a study of *consistent* rules.) Bosmans and Lauwers show that the minimal overlap rule is Lorenz

A parallel conclusion to that of Theorem 1 holds for the CIC family. The proof is parallel too. The only difference is that at each step, the transfer is from one claimant to several others, instead of from several claimants to one other.

Theorem 2 *Let S and S' be two CIC rules, associated with $H \equiv (F, G)$ and $H' \equiv (F', G') \in \bar{\mathcal{H}}^N$ respectively. If, for each $k \in \{1, \ldots, n-1\}$, $F'_k \leq F_k$, then S' Lorenz dominates S. Equivalently, it suffices that for each $k \in \{1, \ldots, n-1\}$, $G'_k \leq G_k$.*

5 Lorenz comparisons of consistent rules

Next, we turn to the family of rules satisfying the following requirement. Consider a rule and apply it to some problem. Then, imagine that a group of claimants receive their awards and leave. Reassess the situation at this point. It can be seen as the problem of allocating among the remaining claimants the sum of the amounts initially awarded to them. Reapply the rule to solve this problem. The requirement is that the rule should assign to each of them the same amount as it did initially. It expresses the robustness of the rule under partial implementation, (when this term is used as in common language, instead of being given its technical meaning).

For a formal statement, one has to generalize the model so as to allow variations in the population of claimants. Let there be an infinite set of "potential" claimants, indexed by the natural numbers, $\mathbb{N}$. In each given problem, however, only a finite number of them are present. Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$. A (claims) **problem** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$, where $N \in \mathcal{N}$, such that $\sum_N c_i \geq E$. A **rule** is a function defined over $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ consisting of all problems involving some population in $\mathcal{N}$, which associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ a vector in $X(c, E)$.

Consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$, if $x \equiv S(c, E)$, then $x_{N'} = S(c_{N'}, \sum_{N'} x_i)$.

dominated by all rules satisfying these properties; moreover, it is the only rule to be so dominated. All ICI rules satisfy *order preservation*, but not all satisfy the *reasonable lower bound*, the constrained equal losses rule being an obvious example of one that does not. Also, an ICI rule satisfies *order preservation under claims variations* only for some choices of the function specifying the breakpoints. Finally an ICI rule satisfies *limited consistency* only if its components pertaining to different populations are appropriately related.

It follows from the definition of a rule that the pair $(c_{N'}, \sum_{N'} x_i)$ is a well-defined problem, so it is meaningful to apply S to it.

Given $N \in \mathcal{N}$, we refer to the restriction of a rule to the subdomain $\mathcal{C}^N$ as its "N-component". What we call an "ICI rule", or a "CIC rule", is now a rule whose components are rules as defined in Subsections 3.2 and 3.3 respectively. A characterization of the family of ICI rules that are *consistent* is available (Thomson, 2007a). Let $\boldsymbol{e} : \mathbb{R} \to \mathbb{R}$ be the identity function. Let $\boldsymbol{\Gamma}$ be the class of nowhere decreasing functions $\gamma \colon \mathbb{R}_+ \to \mathbb{R}_+$, such that $\gamma(0) = 0$ and the function $e - \gamma$ is also nowhere decreasing. (These requirements imply that γ is continuous.) To each $\gamma \in \Gamma$ we associate the ICI rule S^γ defined by setting $F(c) \equiv (n\gamma(\tilde{c}_1), \gamma(\tilde{c}_1) + (n-1)\gamma(\tilde{c}_2), \ldots, \gamma(\tilde{c}_1) + \gamma(\tilde{c}_2) + \cdots + \gamma(\tilde{c}_{n-2}) + 2\gamma(\tilde{c}_{n-1}))$, $\tilde{c}$ being the vector obtained from c by rewriting its coordinates in non-decreasing order, and $G(c)$ being obtained from $F(c)$ by means of the ICI relations. This family of rules—we call them **ICI* rules**—includes the constrained equal awards, constrained equal losses, and Talmud rules. In fact, it contains the entire Moreno-Ternero–Villar (2006a) family. (The minimal overlap rule is not *consistent* however.)

The form taken for ICI* rules by the sufficiency criterion of Theorem 1 is particularly simple. Moreover, it becomes necessary.

Theorem 3 *Let γ and $\gamma' \in \Gamma$. Then, S^γ Lorenz dominates $S^{\gamma'}$ if and only if $\gamma \geq \gamma'$.*

A characterization of the *consistent* members of the CIC family is available too (Thomson, 2007a). These rules—which we call **CIC* rules**—are also indexed by the members of Γ: To each $\gamma \in \Gamma$ we associate the CIC rule R^γ defined by setting $F(c) \equiv (\gamma(\tilde{c}_n) - \gamma(\tilde{c}_{n-1}),\ \gamma(\tilde{c}_n) + \gamma(\tilde{c}_{n-1}) - 2\gamma(\tilde{c}_{n-2}),\ \ldots,\ \gamma(\tilde{c}_n) + \gamma(\tilde{c}_{n-1}) + \cdots + \gamma(\tilde{c}_2) - (n-1)\gamma(\tilde{c}_1))$, $G(c)$ being obtained from $F(c)$ by means of the CIC relations. We have the following counterpart of Theorem 3. We omit the proof, as it is very similar.

Theorem 4 *Let γ and $\gamma' \in \Gamma$. Then, R^γ Lorenz dominates $R^{\gamma'}$ if and only if for each pair $\{a, b\} \in \mathbb{R}_+$ with $a < b$, $\gamma(b) - \gamma(a) \leq \gamma'(b) - \gamma'(a)$.*

More can be said about Lorenz domination within the class of *consistent* rules. Since *consistency* relates the components of a rule across populations, one may hope that a behavior of a rule for low cardinalities would extend to higher cardinalities. This observation underlies the following definition. A property of a rule is **lifted** if whenever a rule satisfies it in the two-claimant case and is *consistent*, then it satisfies it in general (Hokari and Thomson, 2000). There is a practical advantage to a property being lifted: to prove that a *consistent* rule satisfies it, it suffices to prove that it does so in the usually less cumbersome, often trivial, case of two claimants.

Some properties are not lifted on their own but they are lifted if the rule satisfies some additional property (properties). **Resource monotonicity**, which says that for each claims vector, if the endowment increases, each claimant should receive at least as much as he did initially, has been most helpful in assisting the lifting of properties (again, see Hokari and Thomson, 2000).

We propose here to extend the lifting and assisted lifting ideas to order relations on pairs of rules. Given an order $\succeq$ on the space of rules, the **order is lifted** if for each pair S, S' of rules, if $S \succeq S'$ in the two-claimant case and both rules are *consistent*, then $S \succeq S'$ in general. Our next theorem describes circumstances under which the Lorenz order is lifted. In fact, we only require rules to satisfy **bilateral consistency**, the property obtained from *consistency* by restricting the group of remaining claimants to be of size two.

Theorem 5 *Let S and S' be two rules satisfying* order preservation of awards *in the two-claimant case,* resource monotonicity *in the two-claimant case, and* bilateral consistency*. Then, if S Lorenz dominates S' in the two-claimant case, in fact S Lorenz dominates S' in general.*

Next, we explain how to obtain Lorenz rankings of various rules in the two-claimant case. The **path of awards of a rule for a claims vector** is the locus of the awards vector it chooses as the endowment varies from 0 to the sum of the claims. In the two-claimant case, $S \succeq_L S'$ if for each $N \equiv \{i, j\}$, each $c \in \mathbb{R}^N_+$, and each $0 \leq E \leq \sum c_i$, $S(c, E)$ is at least as close as $S'(c, E)$ to the intersection of the line of equation $t_i + t_j = E$ with the 45° line. For rules satisfying *order preservation of awards*, supposing $c_i < c_j$ and measuring c_i and claimant i's award on the horizontal axis, this means that the path of awards of S for c lies to the southeast of the path of S' for c (Figure 1a).

Since many rules are *bilaterally consistent* (in fact, many rules are *consistent*), we will derive a number of Lorenz rankings as corollaries of Theorem 5. Some pertain to rules not introduced yet. First is the rule for which awards are proportional to claims:

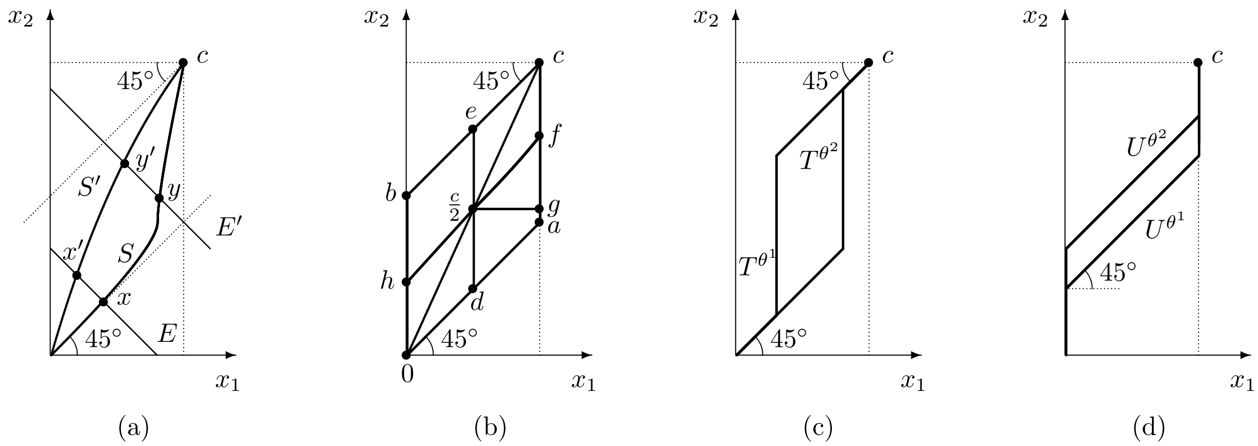

Figure 1: Illustrating Corollary 2. Let $N \equiv \{1, 2\}$, and $c \in \mathbb{R}^N_+$ be such that $c_1 < c_2$. (a) If S Lorenz dominates S' and both rules satisfy *order preservation of awards*, the path of S for c lies to the southeast of the path of S' for c. (b) Paths of awards of several standard rules. (c) Paths of two members of the family $\{T^\theta\}_{\theta\in[0,1]}$. (d) Paths of two members of the family $\{U^\theta\}_{\theta\in[0,1]}$.

Proportional rule, *P*: For each $(c, E) \in \mathcal{C}^N$,

$$P(c, E) \equiv \lambda c,$$

where $\lambda \in \mathbb{R}_+$ is chosen so as to achieve efficiency.

Another rule is defined by applying the constrained equal awards rule "twice", using the half-claims instead of the claims themselves in the formula (for the Talmud rule, there is a switch from the constrained equal awards formula to the constrained equal losses formula, instead of the same formula being used twice).

Piniles' rule, *Pin* (Piniles, 1861) : For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$,

$$Pin_i(c, E) \equiv \begin{cases} \min\{\frac{c_i}{2}, \lambda\} & \text{if } E \leq \sum \frac{c_j}{2}, \\ \frac{c_i}{2} + \min\{\frac{c_i}{2}, \lambda\} & \text{otherwise,} \end{cases}$$

where in each case, $\lambda \in \mathbb{R}_+$ is chosen so as to achieve efficiency.

We refer to the rule defined in a similar way but instead applying the constrained equal losses formula twice, as the **reverse of Piniles' rule**. We omit the formal definition. Like the constrained equal awards rule, our final rule is motivated by egalitarian objectives, but under tighter constraints:

Constrained egalitarian rule, *CE* (Chun, Schummer, and Thomson, 2001) : For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$,

$$CE_i(c, E) \equiv \begin{cases} \min\{\frac{c_i}{2}, \lambda\} & \text{if } E \leq \sum \frac{c_j}{2}, \\ \max\{\frac{c_i}{2}, \min\{c_i, \lambda\}\} & \text{otherwise,} \end{cases}$$

where in each case, $\lambda \in \mathbb{R}_+$ is chosen so as to achieve efficiency.

The paths of several rules are illustrated in Figure 1b for $N \equiv \{1, 2\}$ and $c \in \mathbb{R}^N_+$ with $c_1 < c_2$. Using the notation bro.seg$[x^1, \ldots, x^k]$ to denote the broken line segment connecting these points in that order, the path of the constrained equal awards rule is bro.seg$[0, a, c]$. It is closer to the 45° line than the path of any rule. Because the constrained equal awards rule satisfies the hypotheses of Theorem 5, it Lorenz dominates for any number of claimants all *bilaterally consistent* rules satisfying these hypotheses too.

In Figure 1b the path of the constrained equal losses rule is bro.seg$[0, b, c]$. It lies further away from the 45° line than the path of any rule satisfying *order preservation*. By Theorem 5, this rule is Lorenz dominated for any number of claimants by all *bilaterally consistent* rules satisfying the hypotheses of Theorem 5 and *order preservation of losses*.[11]

We do not want to emphasize these two conclusions however since more general statements can easily be proved, as we have already pointed out in our discussion of Theorem 1. The following Lorenz comparisons are the interesting implications of Theorem 5:

Corollary 2 *(a) Piniles' rule Lorenz dominates the Talmud, proportional, and reverse Talmud rules.*

(b) The constrained egalitarian rule Lorenz dominates Piniles' rule.

(c) The Talmud, proportional, and reverse Talmud rules Lorenz dominate the reverse of Piniles' rule.

(d) Let $\theta^1, \theta^2 \in [0, 1]$ *be such that* $\theta^1 \leq \theta^2$. *Then* T^{θ^2} *Lorenz dominates* T^{θ^1}.

(e) Let $\theta^1, \theta^2 \in [0, 1]$ *be such that* $\theta^1 \leq \theta^2$. *Then* U^{θ^1} *Lorenz dominates* U^{θ^2}.

Proof: The proof that all of the rules mentioned in the corollary satisfy the properties of Theorem 5 is direct and we omit it. (Some of these facts are already noted by Young, 1987, or Thomson, 2007c.) Thus, to prove that one of them, say S, Lorenz dominates some other one, say S', it suffices to check that S Lorenz dominates S' for two claimants: setting $N \equiv \{1, 2\}$, and $c \in \mathbb{R}^N_+$ such that $c_1 < c_2$, this means that the path of S for c should lie to the southeast of the path of S' for c (Figure 1a). Figures 1b,c,d depict the

[11] It would not suffice to restrict attention to the class of rules satisfying *order preservation of losses*. To see this, consider the claims vector $c \equiv (3, 4)$ and the endowment $E = 3$. Then, $CEL(c, E) = (1, 2) \succ_L (3, 0)$.

paths of the rules listed in the corollary. We leave to the reader the easy inspection that they are geometrically related as we assert.

• The path of Piniles' rule is bro.seg$[0, d, \frac{c}{2}, f, c]$ (Figure 1b).

• The path of the reverse of Piniles' rule is bro.seg$[0, h, \frac{c}{2}, e, c]$ (Figure 1b).

• The path of the Talmud rule is bro.seg$[0, d, e, c]$ (Figure 1b).

• The path of the reverse Talmud rule is seg$[0, h, f, c]$ (Figure 1b).

• The path of the proportional rule is seg$[0, c]$ (Figure 1b).

• For the constrained egalitarian rule, two cases have to be distinguished. If $2c_1 < c_2$ (Figure 1b), its path is bro.seg$[0, d, \frac{c}{2}, g, c]$. If $2c_1 > c_2$, let m be the point where the horizontal line of ordinate $\frac{c_2}{2}$ meets the 45° line, and p be the point where this line meets the vertical line of abscissa c_1. Then, the path is bro.seg$[0, d, \frac{c}{2}, m, p, c]$.

• The path of T^θ is bro.seg$[0, \theta(c_1, c_1), (\theta c_1, c_2 - (1-\theta)c_1), c]$ (Figure 1c).

• The path of U^θ is bro.seg$[0, (0, \theta(c_2 - c_1)), (c_1, (1-\theta)c_1 + \theta c_2), c]$ (Figure 1d). □

Bosmans and Lauwers (2007) obtain the first two statements in (a) as well as (c).[12] We noted earlier that Moreno-Ternero and Villar (2006b) provide a direct proof of (d). (So, together with Corollary 1d, we now have three completely different proofs of this statement.)

Consider the requirement on a rule that when the endowment is equal to the half-sum of the claims, it should select the vector of half-claims. This **midpoint property** (Chun, Schummer, and Thomson, 2001) is obviously implied by *self-duality*. In fact, it is considerably weaker as it applies, for each claims vector, to only one endowment, instead of to infinitely many pairs of endowments. Using the logic of Theorem 5, we can show that if two rules S and S' satisfy the *midpoint property* in addition to the hypotheses of this theorem, and S Lorenz dominates S' on the subdomain of two-claimant problems in which the endowment is at most as large as the half-sum of the claims, then S Lorenz dominates S' on this "lower half-domain" for arbitrarily many claimants. A similar statement holds for the "higher half-domain" defined in parallel manner. This is because these half-domains are closed under the reduction operation for rules satisfying the *midpoint*

[12]Bosmans and Lauwers (2007) also offer Lorenz comparisons involving the random arrival rule (O'Neill, 1982) and the adjusted proportional rule (Curiel, Maschler and Tijs, 1987). They are not covered here, since neither rule is a member of the three families we consider (except for the two-claimant case because then, these rules coincide with the Talmud rule).

property. As corollaries, we obtain that on the lower half-domain, the Talmud rule Lorenz dominates the proportional rule, which itself Lorenz dominates the reverse Talmud rule. On the higher half-domain, the reverse Talmud rule Lorenz dominates the proportional rule, which itself Lorenz dominates the Talmud rule. Hougaard and Thorlund-Petersen (2001), Moreno-Ternero and Villar (2006b), and Bosmans and Lauwers (2007) also establish Lorenz rankings on these half-domains.

6 Lorenz comparisons and operators

An operator on the space of rules is a mapping that associates with each rule another one. A systematic investigation of operators is undertaken by Thomson and Yeh (2001). Central are the following three. Given a rule S, the **claims truncation operator** associates with S the rule $\boldsymbol{S^t}$ defined by first truncating claims at the endowment, then applying S: calling $t_i(c, E) \equiv \min\{c_i, E\}$ and $t(c, E) \equiv (t_i(c, E))_{i \in N}$, we have $S^t(c, E) \equiv S(t(c, E), E)$. The **attribution of minimal rights operator** associates with S the rule $\boldsymbol{S^m}$ defined by first assigning to each claimant the difference between the endowment and the sum of the claims of the other claimants if this difference is non-negative and 0 otherwise, then applying S: calling $m_i(c, E) \equiv \max\{E - \sum_{j \in N \setminus \{i\}} c_j, 0\}$ and $m(c, E) \equiv (m_i(c, E))_{i \in N}$, we have $S^m(c, E) \equiv m(c, E) + S(c - m(c, E), E - \sum m_i(c, E))$. The **duality operator** associates with S the rule $\boldsymbol{S^d}$ that treats the endowment as S treats the deficit: $S^d(c, E) \equiv c - S(c, \sum c_i - E)$.

Given an operator p and a rule S, let S^p be the rule obtained by subjecting S to p. Given an order $\succeq$ on the space of rules, **operator *p* preserves the order** if for each pair of rules S, S' such that $S \succeq S'$, then $S^p \succeq S'^p$. It **reverses it** if for each pair of rules S, S' such that $S \succeq S'$, then $S'^p \succeq S^p$.

We introduce separately the **convexity operator** as it takes several arguments. Given a list $(S^k)_{k \in K}$ and a list of non-negative weights for them adding up to one, $(\lambda^k)_{k \in K} \in \Delta^K$, the weighted average of these rules is the rule $\sum \lambda^k S^k$.[13] We say that the **convexity operator preserves an order** $\succeq$ on the space of rules if for each rule S, each pair of lists $(S^k)_{k \in K}$ such that for each $k \in K$, $S^k \succeq S$, and each list $(\lambda^k)_{k \in K} \in \Delta^K$, we have $\sum \lambda^k S^k \succeq S$.

[13] For each problem, the set of awards vectors for it is convex, so this operation is well defined.

The following theorem summarizes our results concerning the preservation of the Lorenz order by operators: one can say that preservation, or reversal, occurs under very general conditions.

Theorem 6 *(a) The claims truncation operator preserves the Lorenz order.*

(b) So does the attribution of minimal rights operator for any two rules satisfying order preservation of awards*.*

(c) The duality operator reverses it for any two rules satisfying order preservation*.*

(d) The convexity operator preserves the Lorenz order within the class of rules satisfying order preservation of awards*.*

Hougaard and Thorlund-Petersen (2001) already state (c).[14]

Given any rule, one may also ask whether it can be Lorenz compared to the rule obtained by subjecting it to a particular operator. The following theorem provides an answer. It involves two properties not introduced yet. **Claims monotonicity** of a rule says that whenever an agent's claim decreases, he should receive at most as much as he did initially. We may be interested in restricting the impact this decrease has on the other agents' awards. In the aggregate, they cannot receive less, but we may require that in fact, each should receive at least as much as he did initially. The requirement is considered by Thomson (2007c) under the name of **others-oriented claims monotonicity**. Generalizing the idea to possible decreases in the claims of several agents at once, we require that each of the other agents should receive at least as much as he did initially (which implies that at least one of the agents whose claim decreases should receive at most as much as he did initially).[15] Let us call this requirement **strong others-oriented**

[14]For completeness and to show how the two parts of *order preservation* are used, we gave an explicit proof. Bosmans and Lauwers (2007) also discuss the reversal of the Lorenz order by the duality operator. The following example shows that *order preservation of losses* is needed. Let $N \equiv \{1, 2\}$ and $c \equiv (1, 2) \in \mathbb{R}^N$. Let S be a rule whose path of awards for c is bro.seg$[(0, 0), (0, \frac{3}{2}), c]$ and whose path coincides with that of CEL for each other claims vector. (This rule is not *claims continuous* but it can be easily modified to satisfy this property). It is clear that $CEL \succeq_L S$, but it is not the case that $S^d \succeq_L CEA = CEL^d$. Indeed, the path of S^d for c is bro.seg$[(0, 0), (1, \frac{1}{2}), c]$. The relevance of *order preservation of awards* is established by using the same example: $CEA \succeq_L S^d$, but it is not the case that $(S^d)^d = S \succeq_L CEL = CEA^d$.

[15]The requirement that each of the agents whose claim decreases should receive at most as much as he did initially is very strong, as it covers the case when one of these agents experiences a very small decrease in his claim. His award should be allowed to increase.

claims monotonicity. Most rules satisfy it. Also, say that two properties are **dual** if whenever a rule satisfies one of them, the dual rule satisfies the other. The *dual of strong others-oriented monotonicity* (we do not attempt to find a name for it) says that if the claims of several agents decrease and the endowment decreases by the sum of the decreases of these claims, then each of the agents whose claims do not change receives at most as much as he did initially. Most rules satisfy this requirement too.

Theorem 7 *(a) Let S be a rule satisfying* order preservation of awards *and* strong others-oriented claims monotonicity*. Then, S is Lorenz dominated by the rule obtained by subjecting S to the claims truncation operator.*

(b) Let S be a rule satisfying order preservation *and* the dual of strong others-oriented claims monotonicity*. Then, S Lorenz dominates the rule obtained by subjecting S to the attribution of minimal rights operator.*

Remark 1: Say that a rule **strictly Lorenz dominates** another one if it Lorenz dominates it and in addition, they are not Lorenz equivalent. All but one of our theorems hold for this notion of domination. The obvious exception is Theorem 6. Two rules may be strictly Lorenz ordered, but coincide when subjected to the claims truncation operator. To illustrate, set $N \equiv \{1, 2\}$ and note that the Talmud rule strictly Lorenz dominates the constrained equal losses rule. However, the Talmud rule is invariant under the claims truncation operator, whereas the image of the constrained equal losses rule under that operator is the Talmud rule itself.

Remark 2: Throughout, we have focused on Lorenz comparisons of vectors of awards, but one may be interested in Lorenz comparisons of vectors of losses (Hougaard and Thorlund-Petersen, 2001). Any result pertaining to awards can be converted by duality to one pertaining to losses.

7 Concluding comments

A wide range of properties have been formulated to evaluate rules. The results described here provide us with another tool in this evaluation process. The Lorenz order on the space of vectors is incomplete—that is, two awards vectors for a given problem may not be comparable in this order—yet we have found a large number of pairs of rules that can be Lorenz ordered uniformly, independently of the number of claimants, the values of their claims, and the endowment. More importantly, we hope that the general techniques we developed to obtain these results, which exploit the facts that (i) rules can be structured into families, (ii) under mild conditions, the Lorenz order is lifted by *bilateral consistency*, and (iii) also under mild conditions, the Lorenz order is preserved, or reversed, by certain operators, reveal what really underlies these rankings, and thus that they will help obtain others.

Appendix

Proof of Theorem 1

Let x and $x' \in \mathbb{R}^N_+$ be such that $\sum x_i = \sum x'_i$. We will use the fact that $x' \succeq_L x$ if and only if x' can be obtained from x by means of a sequence of transfers from one (or several) claimant(s) to one (or several) claimant(s) receiving less, provided no transfer reverses the way in which claimants are ordered.

Let $c \in \mathbb{R}^N_+$. If all coordinates of c differ, the passage from H to H' can be described as the composition of increases, for some $k \in \{1, \ldots, n-1\}$, of the k-th coordinate of the vector $F(c)$. Our first lemma states that if one coordinate of $F(c)$ increases, then for each $E \in [0, \sum c_i]$, a Lorenz improvement occurs in the awards vectors chosen for (c, E). It also covers the case when several coordinates of c are equal; then, successive coordinates of $F(c)$ increase simultaneously.

Lemma 3 *Let S and S' be two ICI rules, associated with $H \equiv (F, G)$ and $H' \equiv (F', G') \in \mathcal{H}^N$ respectively. Let $c \in \mathbb{R}^N_+$. If there are $k \in \{1, \ldots, n-1\}$ and $\bar{k} \in \{0, \ldots, n-k-1\}$ such that $F_k(c) = \cdots = F_{k+\bar{k}}(c) < F'_k(c) = \cdots = F'_{k+\bar{k}}(c)$ and for each $\ell \notin \{k, \ldots, k+\bar{k}\}$, $F_\ell(c) = F'_\ell(c)$, then for each $E \in [0, \sum c_i]$, $S'(c, E) \succeq_L S(c, E)$.*

Proof: For each $k = 1, \ldots, n-1$, let $\delta_k(c) \equiv G_k(c) - F_k(c)$. First, we show that the sequence $(\delta_k(c))_{k=1}^{n-1}$ is independent of which ICI rule is considered.[16] Indeed:

$$\begin{array}{rcccccccccccc}
\delta_1(c) & = & -(n-1)c_1 & + & c_2 & + & c_3 & + & \cdots & + & c_{n-1} & + & c_n \\
\delta_2(c) & = & & & -(n-2)c_2 & + & c_3 & + & \cdots & + & c_{n-1} & + & c_n \\
\cdots & = & & & & + & \cdots & + & \cdots & + & c_{n-1} & + & c_n \\
\delta_{n-2}(c) & = & & & & & & & -2c_{n-2} & + & c_{n-1} & + & c_n \\
\delta_{n-1}(c) & = & & & & & & & & & -c_{n-1} & + & c_n
\end{array}$$

[16] In light of this fact, we could just as well have stated the theorem in terms of G and G' instead of F and F'.

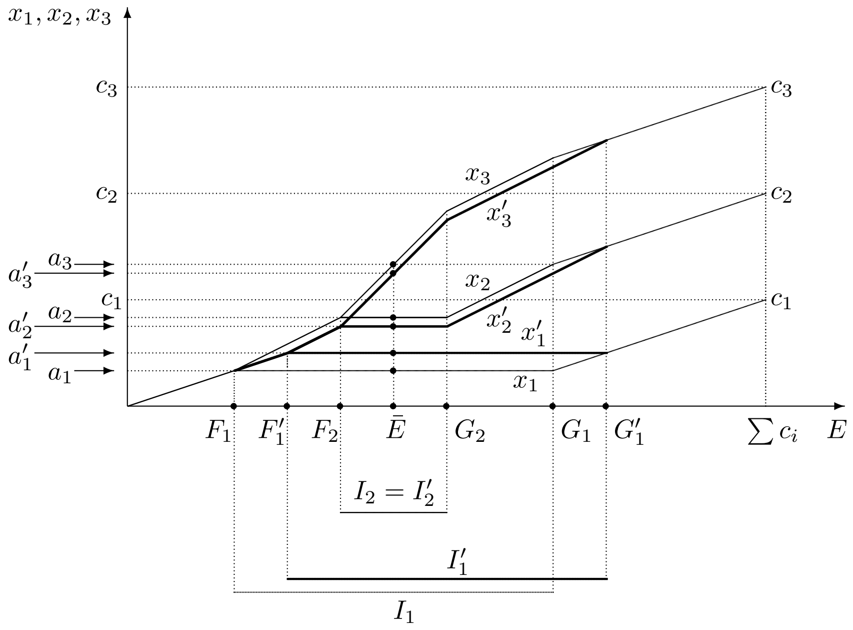

Figure 2: Illustrating Lemma 3. For the claims vector (c_1, c_2, c_3), and starting from the two intervals $I_1 \equiv [F_1, G_1]$ and $I_2 \equiv [F_2, G_2]$ defining an ICI rule, we move I_1 to the right (we drop the argument c to simplify the notation), obtaining $I'_1 \equiv [F'_1, G'_1]$. For values of the endowment in $[0, F_1]$, this does not affect awards. As the endowment increases from F_1 to F'_1, the distribution is made more favorable to claimant 1 at the expense of claimants 2 and 3. The differences between the two distributions remain the same for each endowment in $[F'_1, G_1]$. The two distributions obtained for a typical $\bar{E}$ in that interval are called a and a'. The process is reversed as the endowment increases from G_1 to G'_1, at which point the two distributions are the same. For values of the endowment in $[G'_1, \sum c_i]$, the two distributions remain the same.

Thus, the ICI rule associated with each $(F, G) \in \mathcal{H}^N$ is defined, for each $c \in \mathbb{R}_+$, by the location of nested subintervals of $[0, \sum c_i]$, $I_1(c) \equiv [F_1(c), G_1(c)]$, $I_2(c) \equiv [F_2(c), G_2(c)]$, $\ldots$, $I_{n-1}(c) \equiv [F_{n-1}(c), G_{n-1}(c)]$, of lengths $\delta_1(c)$, $\delta_2(c)$, $\ldots$, $\delta_{n-1}(c)$.

To prove Lemma 3, let us suppose that no two claims are equal, which means that $\bar{k} = 0$ in its statement. Since the calculations are performed for each claims vector separately, we now omit c as an argument of the F_k's and G_k's. Let $k \in \{1, \ldots, n-1\}$, and consider the passage from F_k to $F'_k > F_k$. We show that for each $E \in [0, \sum c_i]$, $x' \equiv S'(c, E)$ is obtained from $x \equiv S(c, E)$ by means of Lorenz improving transfers.

Case 1: $\boldsymbol{E \in [0, F_k]}$. Then $x' = x$ and the conclusion holds trivially.

Case 2: $\boldsymbol{E \in [F_k, F'_k]}$. (This means that claimant k stops partaking in the distribution later for S' than for S.) Let $\Delta \equiv E - F_k$. Under S, this difference is distributed equally among claimants $k+1, \dots, n$. Under S', it is distributed equally among claimants $k, \dots, n$, whereas claimants 1, $\dots$, $k-1$'s awards remain at the values they are given by both rules when $E = F_k$. The differences between the successive partial sums for S' and S, $\sum_{i=1}^{m}[S'_i(c,E) - S_i(c,E)]$, as m runs from 1 to n, are the following: the first $k-1$ differences are 0; then, letting ℓ run from 0 to $n-k$, the $(k+\ell)$-th difference is $\frac{\Delta(n-k-\ell)}{(n-k)(n-k+1)}$. It is important to note for Case 4 that the maximal values of these differences are obtained for $\Delta^* \equiv F'_k - F_k$ (then $E = F'_k$).

Case 3: $\boldsymbol{E \in [F'_k, G_k]}$. Each increment of E in that interval is distributed in the same fashion by both rules. Thus, the differences between the successive partial sums remain what they are under Case 2 for $E = F'_k$, the greatest value of the endowment covered under that case.

Case 4: $\boldsymbol{E \in [G_k, G'_k]}$. (This means that claimant k returns to partake in the distribution later for S' than for S.) Let $\bar{\Delta} \equiv E - G_k$. Under S, this difference is distributed equally among claimants $k, \dots, n$. Under S', it is distributed equally among claimants $k+1, \dots, n$. Starting from an endowment of G_k, the contributions to the differences between the successive partial sums for S' and S of an increase of $\bar{\Delta}$ of the endowment are the following: the first $k-1$ differences are 0; then, letting ℓ run from 0 to $n-k$, the $(k+\ell)$-th term is $\frac{\bar{\Delta}(\ell-n+k)}{(n-k)(n-k+1)}$. The maximal values of these differences are obtained for $\bar{\Delta}^* \equiv G'_k - G_k$ (then $E = G'_k$). At that point and since by the k-th ICI relation, $G'_k - G_k = F'_k - F_k$, the differences are the negative of the differences of the successive partial sums for S' and S reached for $E = F'_k$, so that the successive partial sums are equal for both rules.

Case 5: $\boldsymbol{E \in [G'_k, \sum c_i]}$. As E increases in that interval, each increment is distributed in the same fashion by both rules. The successive partial sums remain equal until E reaches $\sum c_i$.

When several claims are equal, several adjacent parameters are equal for all ICI rules. Then, to pass from one to the other, these parameters have to be increased together. This requires a straightforward adaptation of the calculations just made. If exactly $\bar{k}$ successive parameters are equal, starting with the k-th one, F_k, the transfer is from claimants $k, \dots, k+\bar{k}$ to claimants $k+\bar{k}+1, \dots, n$ for a range of values of the endowment, and later on, for another range of values of the endowment, from claimants $k+\bar{k}+1, \dots, n$ to claimants $k, \dots, k+\bar{k}$. □

Lemma 4 *Under the hypotheses of Theorem 1, one can pass from H to H' by a finite succession of adjustments as described in Lemma 3.*

The proof is simple but not standard and it is notationally cumbersome, so we will first illustrate it by means of examples, supposing initially that no two claims are equal. Again, we drop c as an argument of the F_k's, G_k's, and δ_k's. Given two ICI rules S and S', associated with $H \equiv (F, G)$ and $H' \equiv (F', G') \in \mathcal{H}^N$, if $F \leq F'$, then each interval $I'_k \equiv [F'_k, G'_k]$ for S' is obtained by moving to the right the corresponding interval $I_k \equiv [F_k, G_k]$ for S. (Recall that the intervals have the same length δ_k.) The intervals defining each rule are nested and to pass from S to S', we move intervals in succession, but to remain within the family, we have to respect these nesting constraints. For the two-claimant case, there is a single interval, and there are actually no nesting constraint: one can pass from I_1 to I'_1 in one step.

If $n \geq 3$, more than one step may be needed. Suppose $n = 3$. Then there are two intervals, I_1 and I_2. In each of the examples below, $F_1 = F_2 = 0$ (thus, $S = CEL$) and $G'_1 = G'_2 = \sum c_i$ (thus, $S' = CEA$). This is the case for which the required movements are the largest, and they give us an upper bound on the number of steps. To preserve nesting, we move I_2 to the right, then I_1, then I_2 again, and so on.

Example 1 *(Figure 3a): Let $N \equiv \{1, 2, 3\}$ and $c \equiv (4, 5, 6)$. Then, $\delta_1 = 3$ and $\delta_2 = 1$. The initial positions of the two intervals $[F_1, G_1]$ and $[F_2, G_2]$ are labelled 0. Since their left endpoints are equal at first, we can only move the upper one. The difference $\delta_1 - \delta_2$ of their lengths is equal to 2. It is the maximal amount by which we can move the upper interval. The result of this first move is labelled 1. Once this is done, we can move the lower interval, by a maximal amount also equal to $\delta_1 - \delta_2 = 2$. The result of this second move is labelled 2. We now return to the higher interval. Once again, we can move it by the same amount. The result of this third move is labelled 3. We continue in this manner until the right endpoints of both intervals are $\sum c_i$. Altogether, for the example, we need 13 steps. (The successive values of the left endpoints of the two intervals are $(0, 0)$, $(0, 2)$, $(2, 2)$, $(2, 4)$, $(4, 4)$, $(4, 6)$, $(6, 6)$, $(6, 8)$, $(8, 8)$, $(8, 10)$, $(10, 10)$, $(10, 12)$, $(12, 12)$, $(12, 14)$.)*

Example 2 *(Figure 3b): Let $N \equiv \{1, 2, 3\}$ and $c \equiv (1, 2, 12)$. Then, $\delta_1 = 12$ and $\delta_2 = 10$. The sum of the claims is the same as in Example 1 but we need fewer steps to pass from the constrained equal losses rule to the constrained equal awards rule. The difference $\delta_1 - \delta_2$ is still equal to 2, but because δ_1 is*

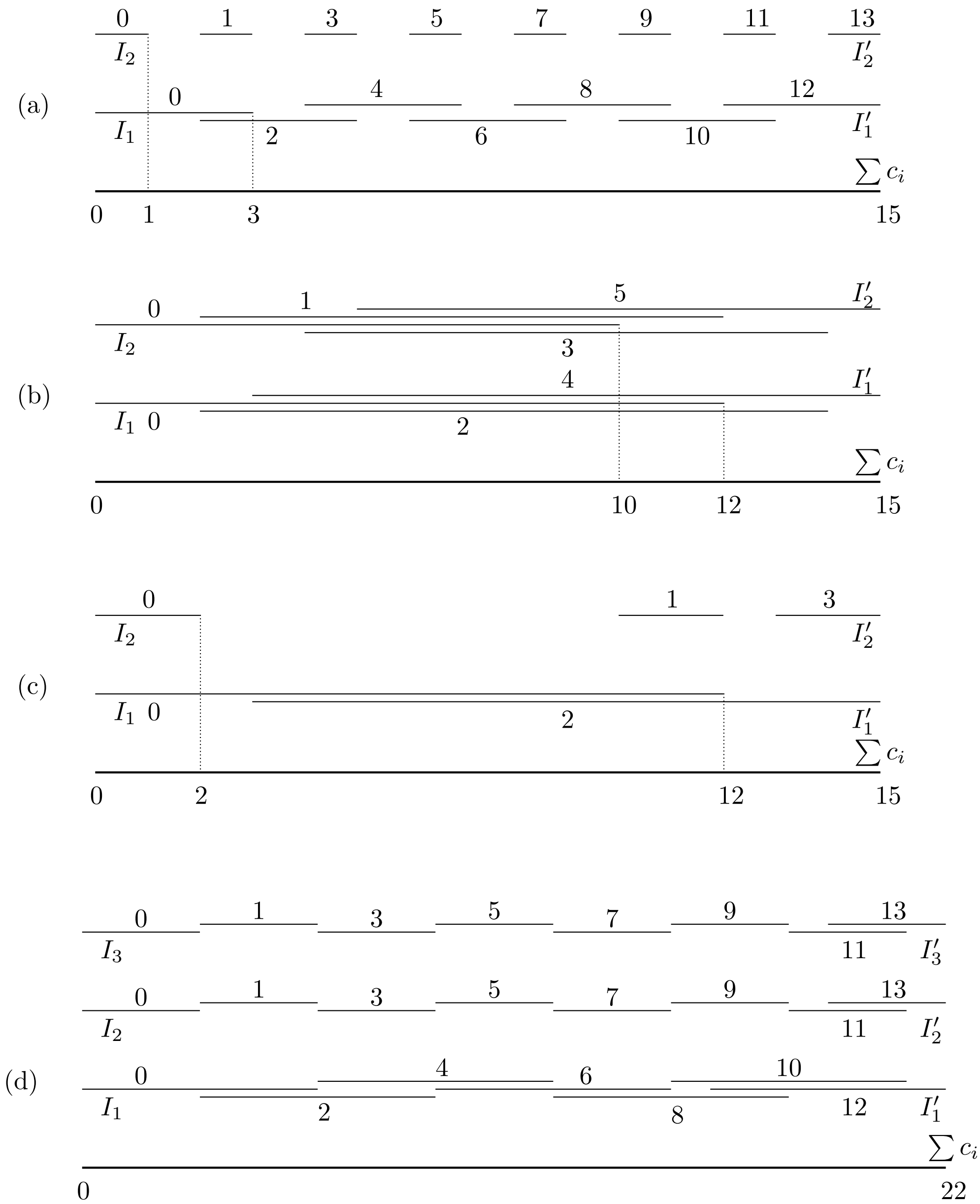

Figure 3: Illustrating Lemma 4. (a) Example 1. (b) Example 2. (c) Example 3. The intervals I_1 and I_2 are represented as horizontal segments at two heights. Their successive positions are numbered, starting with "0". At Step 1, we move the higher interval to the right, labelling its new position "1". At Step 2, we move the lower interval to the right, labelling its new position "2", and so on. (d) For Example 4, there are three intervals, at three different heights, I_1, I_2, and I_3. Since $c_2 = c_3$, I_2 and I_3 have equal lengths, and we have to move them together.

close to $\sum c_i$*, the required move is more limited. (The 5 successive values of the left endpoints of the two intervals are* $(0,0)$*,* $(0,2)$*,* $(2,2)$*,* $(2,4)$*,* $(3,4)$*,* $(3,5)$*.)*

Example 3 *(Figure 3c): Let* $N \equiv \{1,2,3\}$ *and* $c \equiv (1,6,8)$*. Then,* $\delta_1 = 12$ *and* $\delta_2 = 2$*. We have to move the higher interval by a large amount, but because the difference* $\delta_1 - \delta_2$ *is larger than for Example 2, at each step we can move it by a larger amount than we could then. (The 3 successive values of the left endpoints of the two intervals are* $(0,0)$*,* $(0,10)$*,* $(3,10)$*,* $(3,13)$*.)*

These examples show that when claims are all distinct, we can move each interval by the difference between the (positive) lengths of two successive intervals. Since the distance by which we have to move each interval is finite, we need a finite number of steps. The next example is one for which two claims are equal.

Example 4 *(Figure 3d): Let* $N \equiv \{1,2,3,4\}$ *and* $c \equiv (4,5,5,8)$*. Then,* $\delta_1 = 6$ *and* $\delta_2 = \delta_3 = 3$*. Here, two intervals have equal lengths, so we have to move them together. (The 13 successive values of the left endpoints of the three intervals are* $(0,0,0)$*,* $(0,3,3)$*,* $(3,3,3)$*,* $(3,6,6)$*,* $(6,6,6)$*,* $(6,9,9)$*,* $(9,9,9)$*,* $(9,12,12)$*,* $(12,12,12)$*,* $(12,15,15)$*,* $(15,15,15)$*,* $(15,18,18)$*,* $(16,18,18)$*,* $(16,19,19)$*.)*

To prove Lemma 4, we can now generalize the process just described.

Proof: (of Lemma 4) The proof is by induction on the number of claimants. The assertion is true for two claimants as there is only one interval then, and we can move it to its required position in one step. Suppose the assertion is true for $n-1$ claimants. We prove it for n claimants. We undertake a sequence of moves of intervals to the right. Suppose first that all claims are distinct. We move I_{n-1} and I_{n-2} in turn. Moving I_{n-2} by any amount may require moving I_{n-3}, ..., I_1 first but by the induction hypothesis, this requires finitely many steps. If at any step, one of the intervals I_k reaches the position I'_k, we stop for that interval, and the exercise splits into two parts: one of them involves all claimants $\ell > k$ and the other involves all claimants $\ell < k$. We proceed with each group separately but for each of them, by the induction hypothesis, the number of required steps is finite.

The minimal sum of the distance by which we can move I_{n-1} and I_{n-2} during the first two steps if more than two steps are needed is given by the difference $\delta_{n-1} - \delta_{n-2} = 2(c_{n-1} - c_{n-2}) > 0$. At each of the next steps, we can move either one or the other of these intervals by this difference. As the distance by which we have to move each of them is finite, we need finitely many steps. Altogether, we can therefore move all intervals from their initial to their final positions in finitely many steps.

If some claims are equal, we have to move some intervals together, but this can only accelerate the process.

□

Proof: (of Theorem 1) The first part is a direct consequence of Lemmas 3 and 4, and the fact that the Lorenz domination relation is transitive. The second part follows from the first part and the fact that the sequence $(\delta_k)_{k=1}^{n-1} \equiv (G_k - F_k)_{k=1}^{n-1}$ is the same for all members of the family, as established in Lemma 3. □

Proof: (of Theorem 2) We omit the details. The proof involves the observation that each CIC rule can be described by means of nested intervals whose lengths are independent of which member of the family is considered, as was the case for the ICI family. For each $k = 1, \ldots, n-1$, let $\epsilon_k(c) \equiv G_k(c) - F_k(c)$. For each member of the CIC family, the sequence $(\epsilon_k(c))_{k=1}^{n-1}$, which we calculate from the CIC relations, is given by:[17]

$$\begin{array}{rcl} \epsilon_1(c) & = & c_1 + c_2 + \cdots + c_{n-3} + c_{n-2} + 2c_{n-1} \\ \epsilon_2(c) & = & c_1 + c_2 + \cdots + c_{n-3} + 3c_{n-2} \\ \cdots & = & \cdots + \cdots + \cdots + \cdots \\ \epsilon_{n-2}(c) & = & c_1 + (n-1)c_2 \\ \epsilon_{n-1}(c) & = & nc_1 \end{array}$$

□

Proof: (of Theorem 3) The sufficiency part is a consequence of Theorem 1. Conversely, suppose that there is $c_0 \in \mathbb{R}_+$ such that $\gamma(c_0) < \gamma'(c_0)$. Let $N \equiv \{1, 2\}$ and $c \in \mathbb{R}_+^N$ such that $c_1 = c_0$ and $c_2 > c_0$. It is easy to see that the path of S^γ for c lies to the northwest of the path of $S^{\gamma'}$ for c, so in fact, for each $E \in [0, \sum c_i]$, $S^{\gamma'}(c, E) \succeq_L S^\gamma(c, E)$, and because the paths differ, there are values of E for which $S^{\gamma'}(c, E) \succ_L S^\gamma(c, E)$. □

Proof: (Theorem 5) Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Let $x \equiv S(c, E)$ and $x' \equiv S'(c, E)$. To simplify notation, suppose $N \equiv \{1, \ldots, n\}$ and $c_1 \leq \cdots \leq$

[17]We have $\delta_1 + \epsilon_{n-1} = \delta_2 + \epsilon_{n-2} = \cdots = \delta_{n-1} + \epsilon_1 = \sum c_i$.

c_n. Since S and S' satisfy *order preservation of awards* in the two-claimant case and both are *bilaterally consistent*, they satisfy *order preservation of awards* in general, as this property is lifted (Hokari and Thomson, 2000). Thus, $x_1 \leq \cdots \leq x_n$ and $x'_1 \leq \cdots \leq x'_n$. Suppose that x does not Lorenz dominate x'. Then, there is $i^* \in N$ such that $\sum_{\ell=1}^{i^*} x'_\ell > \sum_{\ell=1}^{i^*} x_\ell$. Let i be the smallest index for which such an inequality holds. Obviously, $x'_i > x_i$. Also, $\sum_N x_k = \sum_N x'_k$, and thus, there is $j \in N$ with $j > i$ such that $x'_j < x_j$. Let $N' \equiv \{i, j\}$. Since S is *bilaterally consistent*, $(x_i, x_j) = S(c_i, c_j, x_i + x_j)$. Since S' is *bilaterally consistent*, $(x'_i, x'_j) = S'(c_i, c_j, x'_i + x'_j)$. Let $y \equiv S'(c_i, c_j, x_i + x_j)$. If $x'_i + x'_j \leq x_i + x_j$, then since S' is *resource monotonic* in the two-claimant case, $y_i \geq x'_i > x_i$. If $x'_i + x'_j > x_i + x_j$, then since S' is *resource monotonic* in the two-claimant case, $y_j \leq x'_j < x_j$. Since S and S' satisfy *order preservation of awards*, $x_i \leq x_j$ and $y_i \leq y_j$. In either case, we conclude that $S'(c_i, c_j, x_i + x_j) = (y_i, y_j) \succ_L (x_i, x_j) = S(c_i, c_j, x_i + x_j)$. Thus, it is not true that $S \succeq_L S'$ in the two-claimant case. □

Proof: (of Theorem 6) Let S and S' be two rules such that $S \succeq_L S'$. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$.

(a) We have $S^t(c, E) \equiv S(t(c, E), E) \succeq_L S'(t(c, E), E) \equiv S'^t(c, E)$.

(b) Suppose $N \equiv \{1, \ldots, n\}$ and $c_1 \leq \cdots \leq c_n$. We need to compare $S^m(c, E) = m(c, E) + S(c - m(c, E), E - \sum m_i(c, E))$ and $S'^m(c, E) = m(c, E) + S'(c - m(c, E), E - \sum m_i(c, E))$. Obviously, $m_1(c, E) \leq \cdots \leq m_n(c, E)$ and a simple calculation shows that $c_1 - m_1(c, E) \leq \cdots \leq c_n - m_n(c, E)$ (Thomson, 2007c). Thus, since S and S' satisfy *order preservation of awards*, $S_1(c - m(c, E), E - \sum m_i(c, E)) \leq \cdots \leq S_n(c - m(c, E), E - \sum m_i(c, E))$ and $S'_1(c - m(c, E), E - \sum m_i(c, E)) \leq \cdots \leq S'_n(c - m(c, E), E - \sum m_i(c, E))$. Altogether then, $m_1(c, E) + S_1(c - m(c, E), E - \sum m_i(c, E)) \leq \cdots \leq m_n(c, E) + S_n(c - m(c, E), E - \sum m_i(c, E))$ and similarly, $m_1(c, E) + S'_1(c - m(c, E), E - \sum m_i(c, E)) \leq \cdots \leq m_n(c, E) + S'_n(c - m(c, E), E - \sum m_i(c, E))$. Since $S \succeq_L S'$, then for each $k = 1, \ldots, n$, $\sum_{i=1}^k S_i(c - m(c, E), E - \sum m_i(c, E)) \geq \sum_{i=1}^k S'_k(c - m(c, E), E - \sum m_i(c, E))$. For each $k = 1, \ldots, n$, the k-th partial sums for $S^m(c, E)$ and $S'^m(c, E)$ are obtained from these partial sums by adding the same quantity $\sum_{i=1}^k m_i(c, E)$. Thus, the inequalities between them are preserved and $S^m \succeq_L S'^m$.

(c) Let S and S' be two rules satisfying *order preservation*. Suppose $N \equiv \{1, \ldots, n\}$ and $c_1 \leq \cdots \leq c_n$. We need to show that $S'^d(c, E) \succeq_L S^d(c, E)$, which by definition of the duality operator, means that $[c - S'(c, \sum c_i - E)] \succeq_L [c - S(c, \sum c_i - E)]$. Let $\bar{E} \equiv \sum c_i - E$. Since S and S' satisfy *order preservation of losses*, $c_1 - S_1(c,\bar{E}) \leq \cdots \leq c_n - S_n(c,\bar{E})$ and $c_1 - S'_1(c,\bar{E}) \leq \cdots \leq c_n - S'_n(c,\bar{E})$. Thus, we have to show that for each $k = 1,\ldots,n$, $\sum_{i=1}^{k}[c_i - S'_i(c,\bar{E})] \geq \sum_{i=1}^{k}[c_i - S_i(c,\bar{E})]$. For each $k = 1,\ldots,n$, after canceling from both sides of the k-th inequality in this list the partial sum $\sum_{i=1}^{k} c_i$, we obtain the inequality $\sum_{i=1}^{k} S_i(c,\bar{E}) \geq \sum_{i=1}^{k} S'_i(c,\bar{E})$, which holds because S and S' satisfy *order preservation of awards* and $S \succeq_L S'$.

(d) We omit the obvious proof. □

Proof: (of Theorem 7) (a) Let S be a rule satisfying the hypotheses. Let $N \in \mathcal{N}$, $(c,E) \in \mathcal{C}^N$, $x \equiv S(c,E)$, and $N' \equiv \{i \in N : c_i > E\}$. In the problem $(t(c,E),E)$, the claim of each agent $i \in N'$ is truncated at E. Let $y \equiv S(t(c,E),E)$. By *order preservation of awards*, $x_1 \leq \cdots \leq x_n$. Also, the order of claims is not reversed by truncation. By *order preservation of awards*, $y_1 \leq \cdots \leq y_n$. Thus, the successive sums of ordered coordinates of x and y involve the same successive sets of claimants. If $N' = \emptyset$, there is nothing to prove, so let us assume otherwise, and let $i \in N$ be the agent with the smallest index in N'. By *strong others-oriented claims monotonicity*, for each $j < i$ (there may be no such j), $y_j \geq x_j$. Thus $\frac{\sum_{k \geq i} y_k}{|N'|} \geq \frac{\sum_{k \geq i} x_k}{|N'|}$. By *equal treatment of equals*, which *order preservation* implies, for each pair $k,k' \in N'$, $y_k = y_{k'} \equiv a$. Thus, there is $k \in N'$ such that for each $k' \in N'$ with $k' \leq k$, $y_{k'} = a \geq x_{k'}$, and for each $k' > k$, $y_{k'} = a \leq x_{k'}$. Altogether, for each $k' \leq k$, $y_{k'} > x_{k'}$, and for each $k' > k$, $y_{k'} \leq x_{k'}$. Then, an obvious calculation shows that $y \succeq_L x$.

(b) This statement follows from (a) by similar reasoning. It exploits the fact that after attribution of minimal rights, the revised claims of all agents whose minimal rights are positive are equal. □

References

Alcalde, J., M. Marco, and J. A. Silva, "Bankruptcy problems and the Ibn Ezra's proposal", *Economic Theory* 26 (2005), 103-114.

——, ——, and ——, "The minimal overlap rule revisited", *Social Choice and Welfare*, forthcoming, 2007.

Aumann, R. and M. Maschler, "Game theoretic analysis of a bankruptcy problem from the Talmud", *Journal of Economic Theory* 36 (1985), 195-213.

Bergantiños, G. and L. Méndez-Naya, "Additivity in bankruptcy problems and in allocation problems", *Spanish Economic Review* 3 (2001), 223-229.

Bosmans, K. and L. Lauwers, "Lorenz comparisons of nine rules for the adjudication of conflicting claims", mimeo, 2007.

Chun, Y., J. Schummer, and W. Thomson, "Constrained egalitarianism: A new solution to bankruptcy problems", *Seoul Journal of Economics* 14 (2001), 269-297.

Chun, Y. and W. Thomson, "Convergence under replication of rules to adjudicate conflicting claims", *Games and Economic Behavior* 50 (2005), 129-142.

Curiel, I., M. Maschler and S. H. Tijs, "Bankruptcy games", *Zeitschrift für Operations Research* 31 (1987), A143-A159.

Dominguez, D. and W. Thomson, "A new solution to the problem of adjudicating conflicting claims", *Economic Theory* 28 (2006), 283-307.

Herrero, C. and A. Villar, "The three musketeers: four classical solutions to bankruptcy problems", *Mathematical Social Sciences* 39 (2001), 307-328.

Hokari, T and W. Thomson, "On properties of division rules lifted by bilateral consistency", mimeo 2000, revised 2006, *Journal of Mathematical Economics*, forthcoming.

Hougaard, J. L. and L. P. Østerdal, "Inequality preserving rationing", *Economics Letters* 87 (2005), 355-360.

Hougaard, J. L. and L. Thorlund-Peterson, "Bankruptcy rules, inequality, and uncertainty", mimeo, 2001.

Ju, B.-G. and J. Moreno-Ternero, "Progressivity, inequality reduction and merging-proofness in taxation", mimeo, 2006.

—— and ——, "On the equivalence between progressive taxation and inequality reduction", *Social Choice and Welfare*, forthcoming, 2007.

Kasajima, Y. and R. Velez, "Proportional and non-proportional inequality preserving rationing", mimeo, 2008.

Marchant, T., "Scale invariance and similar invariance conditions for bankruptcy problems", *Social Choice and Welfare* (2008), forthcoming.

Moreno-Ternero, J. and A. Villar, "The Talmud rule and the securement of agents' awards", *Mathematical Social Sciences* 47 (2004), 245-257.

—— and ——, "The TAL-family of rules for bankruptcy problems", *Social Choice and Welfare* 27 (2006a), 231-249.

—— and ——, "On the relative equitability of a family of taxation rules", *Journal of Public Economic Theory* 8 (2006b), 283-291.

O'Neill, B., "A problem of rights arbitration from the Talmud", *Mathematical Social Sciences* 2 (1982), 345-371.

Piniles, H. M., *Darkah shel Torah*, 1861, Forester, Vienna.

Schummer, J. and W. Thomson, "Two derivations of the uniform rule", *Economics Letters* 55 (1997), 333-337.

Thomson, W., Condorcet Lecture, Fourth International Meeting of the Society for Social Choice and Welfare, Alicante, 2000.

——, "Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey", *Mathematical Social Sciences* 45 (2003), 249-297.

——, "Two families of rules for the adjudication of conflicting claims", *Social Choice and Welfare*, forthcoming, 2007a.

——, "Consistent allocation rules", mimeo, 2007b.

——, *How to Divide When there Isn't Enough; from the Talmud to Game Theory*, mimeo, 2007c.

Thomson, W. and C.-H. Yeh, "Operators for the adjudication of conflicting claims", mimeo, 2001, revised 2006, *Journal of Economic Theory*, forthcoming.

Young, P., "On dividing an amount according to individual claims or liabilities", *Mathematics of Operations Research* 12 (1987), 398-414.