Events Concerning Knowledge

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Abstract

Aumann's knowledge operator can transform a measurable event into one that is not measurable. This problem prevents his theorem regarding agreement of subjective probabilities from being extended beyond the restrictive framework of countable information partitions. The problem is averted if agents' information partitions are induced by measurable functions from states of the world to the agents' types.

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1 Introduction

Much research in game theory and its applications (to auctions, for example) has to do with situations in one player may know, or may hold a probability belief, about what another player knows or believes. Harsanyi [1967–68] and subsequent game theorists have developed a Bayesian theory of players' beliefs about other players' beliefs. Aumann [1976, 1999a,b] has developed a model of knowledge about other players' knowledge. In particular, in [1999b], he extends the theory to deal explicitly with both knowledge and belief in a common framework. The research reported here addresses an issue in that theory, concerning measurability of events defined by agents' knowledge of other events.

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The intended interpretation of *knowledge* is that it is obtained in a way that is objective in some sense. *Certainty,* which is belief with subjective probability 1, is a concept that is related to, but distinct from, knowledge. To understand the distinction, consider how someone might become certain of the color of a surface. Someone with normal vision might know that he is in a room with a red wall, for example, because he sees it in normal light. In contrast, a color-blind person can be told independently by many others with normal vision that the wall is red, and he may have conclusive reasons to trust them completely, but he can only be certain that the wall is red. He cannot know that it is red because, although he discounts this possibility completely, there is nevertheless a *possible* state of the world in which the wall is green (which color he cannot distinguish by sight from red) and in which his informants are all telling him that it is red.[1]

For many purposes, the distinction between knowledge and certainty is immaterial to game-theoretic analysis. Consider, for example, the following situation.

> Hussein (former dictator of Iraq) is hiding from Casey (commander of invading army) in location L. Hussein believes that either Casey knows that he is hiding in L but is waiting to capture him at a moment of maximum strategic impact, or else Casey is completely ignorant. He wants to move to a new hiding place (a risky activity) if Casey knows that he is hiding in L, but to stay safely in place otherwise.

Now, suppose that the two occurrences of 'Casey knows that he is hiding in L' in the preceding account are both changed to 'Casey is certain that he is hiding in L'. Would that amendment lead to any change in the game-theoretic analysis of whether Hussein's equilibrium strategy is to change hiding places or to stay put? The intuitive answer is that it would not. Although there may be a genuine distinction between knowledge and certainty, then, the distinction is immaterial for game-theoretic analysis.

Whether or not it makes a difference whether an *analysis* is carried out in terms of knowledge or certainty, though, there are instances in which the distinction between knowledge and certainty seems crucial for the *formulation* of a game. Suppose, for example, that one were providing a game theoretic analysis of the rules of evidence in a legal system. Consider a celebrated case.

> In 1911, da Vinci's *Mona Lisa* was stolen from the Louvre Museum. Guillaume Apollinaire was arrested as a suspect; he implicated Pablo Picasso, but both were eventually dropped as possible culprits. Subsequently, Italian police arrested a

[1] There is a philosophical argument (introduced by Hanson [1958]) that observation is "theory laden," which suggests that there is no difference in principle between a theory of why one's eyes are not deceiving him and a theory of why one's informants are not deceiving him. That is, on this philosophical view, there is no genuine distinction between knowledge or certainty (or perhaps, equivalently for purposes in game theory, there is a conceptual distinction but it is moot because no one can ever know anything). Regardless of the soundness of this view as a starting point for a theory of perception, however, its adoption in game theory would be highly inconvenient. Everyone recognizes intuitively the difference between seeing something with one's own eyes and hearing about it, even from a highly trusted informant who has no incentive for misrepresentation.

man named Vincenzo Perugia, who had brought the *Mona Lisa* to an antiques dealer to sell it. Perugia, while found guilty, served only a few months in jail. [2]

Now, suppose that Appolinaire had testified that he believed with certainty that Picasso had stolen the painting. Even if that testimony were sincere, it could not have convicted Picasso because subjective belief—even with probability 1—is not admissible evidence. However, the dealer's testimony that he actually saw Perugia walk into his shop with the painting was admissible.

Under U.S. law, even the subjective certainty of a scientific expert is not admissible. "The subject of an expert's testimony must be 'scientific ... [*sic.*] knowledge.' ... [T]he word 'knowledge' connotes more than subjective belief or unsupported speculation."[3] Thus, to provide a game-theoretic analysis of the incentive effect of litigation, it is essential to take account of the distinction between knowledge and certainty.

Arguably, the requirement to distinguish between knowledge versus certainty in modeling (or designing) a legal system reflects a crucial distinction for game theory in general. Knowledge of an event is exogenous or is attained by utilizing some exogenous technology such as a scientific experiment. Any agent in the same position, or utilizing the same technology in the same way, would have or acquire identical knowledge. (Color vision is a sensory technology, as well as being part of an agent's belief system.) In contrast, certainty is an extremely strongly held belief—a property of the agent, not a relation between the agent and his environment—that is endogenously formed rather than resulting from the agent's exogenous situation or being an exogenously specified result of engaging in a deliberate activity. Consider that pure-strategy Nash equilibrium in a noncooperative game is tantamount to mutual certainty of the (rational) players' actions.[4] Nevertheless, because certainty is endogenous, there can be multiple Nash equilibriums. It will not do for the rules or the outcomes of a game to be defined so ambiguously. In fact, to specify a game in terms of its endogenous outcome would risk logical circularity. Therefore, where a game must be specified in terms of an epistemic concept, agents' knowledge rather than their certainty must be the basis on which it is done.

The plan of the paper is as follows. In section 2, a formal, single-agent model of knowledge and belief will be formulated. In section 3, that model will be parametrized in such a way that, for some measurable event, the event that it is known to the agent cannot have a probability because it is not measurable. In that parametrization, knowledge will be represented in terms of an information partition, the events in which are measurable events. That is, the measurability problem occurs although the specification is not problematic in terms of Aumann's [1999b] "semantic" model of knowledge and belief. A detailed comparison with Aumann's model is provided in section 4. In section 5, Aumann's [1976] common-knowledge partition is shown to be subject to a measurability problem related to that exhibited in section 3. In section 6, another respresentation of knowledge will be introduced: the information partition on the space of *states of the world* is induced by a measurable function

[2] *Time* magazine, `http://www.time.com/time/2007/crimes/2.html,` accessed on 2008.06.10.

[3] *Daubert v. Merrell Dow Pharmaceuticals,* 509 U.S. 579 (1993)

[4] Aumann and Brandenburger [1995].

from that space to another measurable space of *knowledge states* or *types*. That is, the agent's type is a random variable, and the elements of the information partition are the inverse images of the types. It is proved that, in this representation, the event that an agent knows a measurable event is also measurable (with respect to the completion of the agent's prior-probability measure). This result is noteworthy because knowledge states (essentially the analogue of Harsanyi's agent types) have been thought to be just a convenient way of defining an information partition, but are now shown to play a crucial role in guaranteeing that the specification of the game satisfies a necessary condition for agents' expected utilities from a strategy profile to be defined. In section 6, the common-knowledge partition of a group of agents whose information partitions are induced by measurable type functions is shown to consist of events that are measurable with respect to all of their prior-probability measures.

2 Events regarding knowledge and certainty

Consider an agent whose *prior beliefs* are specified by a probability space $(\Omega, \mathcal{B}, P)$, where $\mathcal{B}$ is the Borel σ-algebra on Ω and $P : \mathcal{B} \to [0, 1]$ is a countably additive probability measure. It is assumed that[5]

$$\begin{array}{l}\Omega \text{ is a complete, separable metric space.}\\ \mathcal{B} \text{ is the } \sigma\text{-algebra of Borel subsets of } \Omega.\end{array} \tag{1}$$

The agent's *information partition* Π satisfies

$$\Pi \subseteq \mathcal{B} \setminus \{\emptyset\} \text{ and } \forall X \in \Pi\ \forall Y \in \Pi\ [X = Y \text{ or } X \cap Y = \emptyset] \text{ and } \bigcup \Pi = \Omega. \tag{2}$$

That is, the elements of Π are measurable sets that are nonempty and disjoint, and $\bigcup \Pi = \Omega$. Let the variable ω range over Ω, and for each ω, let $\pi(\omega)$ be the unique element of Π to which ω belongs. ($\pi(\omega)$ is called the agent's *information set* at ω.) Define the σ-algebra $B_\Pi \subseteq \mathcal{B}$ by

$$\mathcal{B}_\Pi = \{X \mid X \in \mathcal{B} \text{ and } \forall \omega[\omega \in X \iff \pi(\omega) \subseteq X]\}. \tag{3}$$

Note that $\Pi \subseteq \mathcal{B}_\Pi$.

The agent's *posterior beliefs* are specified by a regular conditional probability function $p : \mathcal{B} \times \Omega \to [0, 1]$. Such a function satisfies the following conditions.

$$\text{For every } \omega,\ p_\omega(X) = p(X, \omega) \text{ defines a Borel probablity measure.} \tag{4}$$

$$\text{For every } X \in \Omega,\ p_X(\omega) = p(X, \omega) \text{ is } B_\Pi\text{-measurable.} \tag{5}$$

$$p_\omega(\pi(\omega)) = 1 \text{ a.s.} \tag{6}$$

[5] More generally, it can be assumed that $(\Omega, \mathcal{B}, P)$ is isomorphic to a Borel subset of a space satisfying this assumption.

$$\text{For every } X \in \mathcal{B} \text{ and } Y \in \mathcal{B}_\Pi,\ \int_Y p_X(\omega)\, dP = P(X \cap Y). \tag{7}$$

Consider an event $X \in \mathcal{B}$. (*Event* will be used in this paper mean an arbitrary subset of Ω. This contrasts with the usual usage, which implies measurability.) That the agent knows X is also an event, as is the event that the agent is certain of X. Let $K(X)$ and $C(X)$ denote the knowledge and certainty events respectively. Here, and henceforth in this paper, *event* means simply a subset of Ω. It does not necessarily mean (as in general mathematical usage) that the subset is a Borel set or an element of some other σ-algebra. Indeed, to find a sufficient condition for measurability of $K(X)$ is the major theoretical problem to be addressed here.

Following Aumann [1976], the agent is specified to know an event X when he is in state ω if his information set in ω is a subset of X. That is, all of the states of the world that his information does not exclude from being the true state are in X. Formally,

$$K(X) = \{\omega \mid \pi(\omega) \subseteq X\}. \tag{8}$$

As discussed above, the agent is specified to be certain of X in ω if $p_\omega(X) = 1$. That is,

$$C(X) = \{\omega \mid p_\omega(X) = 1\}. \tag{9}$$

Definitions (8) and (9), along with properties (5)–(7) of the definition of regular conditional probability, imply the following lemma.

Lemma 1 *For every $X \in \mathcal{B}$, $K(X) \subseteq X$ and $C(X) \in \mathcal{B}_\Pi$. For all $X \in \mathcal{B}_\Pi$, $P(X \triangle C(X)) = 0$.*[6]

3 A measurability problem with $K(X)$

Knowledge of a Borel-measurable event is not necessarily a measurable event, even in the weaker (than Borel) sense of being measurable with respect to the completion of some Borel measure. An example of this problem will be constructed momentarily. To prepare for this example, recall what is the *completion* of a measure.

Let 2^Ω denote the power set of Ω, and let $P : \Omega \to [0, 1]$ be a countably additive measure. Then define $P_* : 2^\Omega \to [0, 1]$ by $P_*(X) = \sup\{P(Y) \mid Y \in \mathcal{B} \text{ and } Y \subseteq X\}$ and define $P^* : 2^\Omega \to [0, 1]$ by $P^*(X) = \inf\{Z \mid Z \in \mathcal{B} \text{ and } X \subseteq Z\}$. The set $\mathcal{B}_P = \{X \mid P_*(X) = P^*(X)\}$ is a σ-algebra, the events in which will be called P *measurable.* The restriction of P^* to $\mathcal{B}_P$ is a countably additive measure that coincides with P on $\mathcal{B}$. To simplify notation, P will be considered to be defined on $\mathcal{B}_P$. The measure on this enlarged σ-algebra is called the *completion* of P.[7]

Call a probability measure *nicely unprejudiced* if it satisfies the following conditions.[8]

[6] The symmetric-difference operation on sets is denoted by $\triangle$.

[7] Halmos [1970, Chapters II–III] presents this material in detail.

[8] In Bayesian statistics, a measure satisfying the first condition is typically called *unprejudiced.* Since atoms have infinitely higher prior probability than do other states of the world, nonatomicity also has a flavor of absence of prejudice.

1. For all open U, $P(U) > 0$, and
2. P is nonatomic. (That is, $\forall \omega\ P(\omega) = 0$.)

Lemma 2 *There is an event V such that*

$$\forall P\ [P \text{ is nicely unprejudiced } \Longrightarrow V \text{ is not } P \text{ measurable }]. \tag{10}$$

9

Example 1 *An information partition Π and a Borel-measurable event X, such that*

$$\forall P\ [P \text{ is nicely unprejudiced } \Longrightarrow K(X) \text{ is not } P \text{ measurable }].$$

Let $\Omega = [0,1]$, and let V be an event satisfying (10). Let $X = \{\omega/2 \mid \omega \in V \setminus \{0,1\}\}$. Clearly $X \subseteq (0,1/2)$ and X satisfies (10), since that property is not affected by deletion of a finite set or by a linear transformation. Define Π by

$$\pi(\omega) = \begin{cases} \{\omega, \omega + 1/2\} & \text{if } \omega \in X, \\ \{\omega, \omega - 1/2\} & \text{if } \omega - 1/2 \in X, \\ \{\omega\} & \text{otherwise.} \end{cases} \tag{11}$$

Now it will be shown that, although $[0,1/2]$ is Borel measurable, $K([0,1/2])$ is not P measurable for any P that has a nonatomic part. $K([0,1/2]) = [0,1/2] \setminus X$, and therefore $[0,1/2] \setminus K([0,1/2]) = X$. Since the difference of two P-measurable sets is also P measurable, and since X is not P measurable if P has a nonatomic part, $K([0,1/2])$ cannot be P measurable if P has a nonatomic part. ∎

Example 1 shows that, if Π satisfies (2) and $X \in \mathcal{B}$, nevertheless it is possible that $K(X)$ defined by (8) may not be measurable with respect to any nicely unprejudiced prior belief. To embed the example in the full model of an agent specified in section 2, satisfying all of the conditions (1)–(7), it is sufficient to specify a nicely unprejudiced prior belief P. Parthasarathy [1967, Theorem 8.1] shows that there is a regular conditional probability measure satisfying (4)–(7) that represents the agent's posterior belief.[10]

4 Comparison with Aumann's framework

The framework of section 2 closely follows a single-agent specialization of Aumann [1999a,b]. Now several differences in exposition, and minor differences in specification, are discussed.

[9] Oxtoby [1980, Theorems 5.3, 5.4]. The proof utilizes the Axiom of Choice.

[10] To apply Parthasarathy's theorem, let X and Y both be Ω, let $\mathcal{C} = \mathcal{B}_\Pi$ and let π be the identity mapping.

in [1999a], Aumann defines an information partition in terms of a mapping from the states of the world to an abstract set of *knowledge states.* The information partition is induced by the equivalence relation of mapping to the same knowledge state. The framework of this paper is equivalent in this regard, since information sets can be taken to be the knowledge states, so that π functions as Aumann's mapping from states of the world to knowledge states.

In [1999b, section 12], Aumann defines a *knowledge-belief system.* While the mapping to knowledge states in [1999a] is not assumed to possess any measurability property, the information sets of a knowledge-belief system are assumed to be measurable events as in (2). However, $(\Omega, \mathcal{B})$ is assumed only to be an abstract measurable space, rather than to be derived from a separable, complete metric space.

A knowledge-belief system includes a function π that satisfies conditions (4) that p_ω is a measure for every ω and (5) that p_X is measurable for every X. Aumann assumes that $K(X) \subseteq C(X)$, which is slightly stronger than assumption (6) here. Specifically, (6) and the definitions of K and C imply that $P(K(X) \setminus C(X)) = 0$, while Aumann's assumption can be reformulated as $K(X) \setminus C(X) = \emptyset$. Aumann also assumes that, for every $X \in \mathcal{B}$ and $\alpha \in [0, 1]$, $K(p_X^{-1}(\alpha)) = p_X^{-1}(\alpha)$. This assumption is implied by assumption (5) that p_X is $\mathcal{B}_\Pi$ measurable (which entails that $p_X^{-1}(\alpha) \in \mathcal{B}_\Pi$ because $\{\alpha\}$ is the intersection of countably many open sets) and the definition (8) of K (which entails that $\forall X \in \mathcal{B}_\Pi\ K(X) = X$).

Aumann's knowledge-belief system does not have an analogue of (7), which is the condition of Bayesian consistency between prior and posterior beliefs. Rather, the system specifies only posterior beliefs. If desired, prior beliefs of a single agent could be specified by taking (7) as a definition. However, that approach would have limitations in a multi-agent setting. For example, if it is desired to impose a common-prior assumption (cf. Harsanyi [1967–68]), then it is more straightforward to do so directly than to verify that the implications of the assumption regarding agents' posterior probabilities are satisfied.

In view of this comparison, it is plausible that an analogue of example 1 can be constructed in Aumann's framework. It is conceivable, however, that the strengthening of (6) to $K(X) \subseteq C(X)$ would prevent that measurability problem from arising.

5 Common knowledge

Aumann [1976] formalizes the theory of common knowledge and proves an "agreement theorem" that, if agents' posterior probabilities of an event in some some state of the world are commonly known to them, then the posterior probabilities are identical if the agents share a common prior. Aumann makes an assumption that implies that the information partition of each agent is a countable set of measurable events, each of which has positive prior probility. In this section, it will be shown that a measurability problem related to example 1 can occur otherwise.

Let's begin by reviewing Aumann's results. Consider a finite set I of agents, and let each agent i have information partition Π_i. Let Π_I be the *common-knowledge partition*, which isthe finest partition that is as coarse as each Π_i, and let π_I^∞ denote the function mapping each state of the world to the information set in Π_I that contains it. Define $K_I(X) = \bigcap_{i \in I} K_i(X)$. The event that X is common knowledge is $\bigcap_{n \in \mathbb{N}} K_I^n(X)$. Denote this event by $K_I^\infty(X)$. Aumann sketches an argument that, for all ω,

$$\omega \in K_I^\infty(X) \Longleftrightarrow \pi_I^\infty(\omega) \subseteq X. \tag{12}$$

The "agreement theorem" is proved by calculating an integral (with respect to the agents' common prior probability measure) over $\pi_I^\infty(\omega)$, which has positive prior measure under the countability assumption. A minor problem with generalizing the theorem to uncountable information partitions is that this event may have zero prior measure. It is straightforward to reformulate the theorem to address that problem. A more fundamental problem is that, as in the following example, $\pi_I^\infty(\omega)$ may fail to be measurable with respect to a nicely unprejudiced prior.

Example 2 *A common-knowledge partition having a non-measurable event.*

Consider two agents. Let Π_1 be the information partition constructed in example 1. (Recall that the construction is based on an event $X \subseteq (0, 1/2)$ that is not measurable with respect to any nicely unprejudiced measure.) Let $\Pi_2 = \{[1/2, 1]\} \cup \{\{\omega\} \mid \omega < 1/2\}$. Both agents' information partitions consist solely of events in $\mathcal{B}$. However, $\Pi_I = \{[1/2, 1] \cup X\} \cup \{\{\omega\} \mid \omega < 1/2 \text{ and } \omega \notin X\}$. Since $\pi_I^\infty(1)$ is the disjoint union of a Borel set and X, it is not measurable with respect to any nicely unprejudiced measure. ∎

6 Measurable assignment of knowledge states induces measurable knowledge events

The measurability problem identified in example 1 does not occur (for any prior probability measure), if the agent's information partition is induced by a Borel-measurable function $\tau : \Omega \to T$, where each T is a separable, complete metric space of knowledge states (or *types*).[11]

Assume that

$$T \text{ is a separable, complete metric space, and } \mathcal{T} \text{ is the Borel } \sigma\text{-algebra on } T. \tag{13}$$

Let $\mathcal{A}$ denote the Borel σ-algebra on $\Omega \times T$. Define a subset X of either Ω or Θ to be *analytic* if it is the projection of a set $Y \in \mathcal{A}$. Define a subset X of either Ω or Θ to be *universally measurable* if, for every measure μ, $\mu_*(X) = \mu^*(X)$.

Lemma 3 *A countable union of analytic sets is analytic. Every analytic set is universally measurable. The complement of a universally measurable set is universally measurable.*[12]

[11] Π is induced by τ if $\Pi = \{\tau^{-1}(t) \mid t \in T\} \setminus \{\emptyset\}$.

[12] The first assertion is proved in Bertsekas and Shreve [1978, Corollary 7.35.2]. The second is proved in Bertsekas and Shreve [1978, Corollary 7.42.1]. The third is immediate from the definition of measurability with respect to μ.

Lemma 4 *If $\tau : \Omega \to T$ is Borel measurable, and $X \subseteq \Omega$ and $Y \subseteq \Theta$ are analytic, then $\tau(X)$ and $\tau^{-1}(Y)$ are analytic.*[13]

Theorem 1 *If Π is the information partition induced by a Borel-measurable function $\tau : \Omega \to T$, where T is a separable, complete metric space, then for every $X \in \mathcal{B}$, $K(X)$ is universally measurable.*

Proof $K(X) = \{\omega \mid \tau(\omega) \notin \tau(\Omega \setminus X) = \Omega \setminus \tau^{-1}(\tau(\Omega \setminus X))$. Event $\tau^{-1}(\tau(\Omega \setminus X))$ is analytic by lemma 4, so it is univerally measurable by lemma 3. Therefore $K(X) = \Omega \setminus \tau^{-1}(\tau(\Omega \setminus X))$ is universally measurable, again by lemma 3. ∎

7 Events in the common-knowledge partition are analytic

The measurability problem in example 2 is also averted, if each agent's information partition is induced by a Borel-measurable type function $\tau_i : \Omega \to T_i$ (where each T_i is a separable complete metric space. The key to proving this fact is stated in terms of a function $\pi_I : 2^\Omega \to 2^\Omega$ defined by $\pi_I(Y) = \bigcup_{i \in I} \bigcup_{\omega \in Y} \pi_i(\omega)$.

Lemma 5 *The function $\pi_I^\infty : \Omega \to 2^\Omega$, which maps each state of the world to the event in the common-knowledge partition that contains it, satisfies*

$$\pi_I^\infty(\omega) = \bigcup_{n \in \mathbb{N}} \pi_I^n(\{\omega\}).^{14} \tag{14}$$

Theorem 2 *For every ω, $\pi_I^\infty(\omega)$ is analytic, and thus universally measurable.*

Proof For an arbitrary event X, $\pi_I(X) = \bigcup_{i \in I} \tau_i^{-1}(\tau_i(X))$. Therefore, by lemma 3, $\pi_I(X)$ is analytic if X is analytic. By induction on $n \in \mathbb{N}$, then, $\pi_I^n(\{\omega\})$ is analytic. Therefore, by lemma 3 and lemma 5, $\pi_I^\infty(\omega)$ is analytic, and thus universally measurable. ∎

[13] Bertsekas and Shreve [1978, Proposition 7.40].

[14] This equation formalizes the conclusion of the paragraph analyzing *reachable from* ω, in Aumann [1976, p. 1237].

References

Robert Aumann and Adam Brandenburger. Epistemic conditions for Nash equilibrium. *International Journal of Game Theory*, 63:1161–1180, 1995.

Robert J. Aumann. Agreeing to disagree. *Annals of Statistics*, 4:1236–1239, 1976.

Robert J. Aumann. Interactive epistemology I: Knowledge. *International Journal of Game Theory*, 28:263–300, 1999a.

Robert J. Aumann. Interactive epistemology II: Probability. *International Journal of Game Theory*, 28:301–314, 1999b.

Dimitri P. Bertsekas and Steven E. Shreve. *Stochastic Optimal Control: The Discrete-Time Case*. Academic Press, 1978. URL `http://web.mit.edu/dimitrib/www/soc.html`.

Paul R. Halmos. *Measure Theory*. Springer-Verlag, 1970.

Norwood Russell Hanson. *Patterns of Discovery*. Cambridge University Press, 1958.

John C. Harsanyi. Games with incomplete information played by Bayesian players. *Management Science*, 14:159–182, 320–334, 486–502, 1967–68.

John C. Oxtoby. *Measure and Category*. Springer-Verlag, second edition, 1980.

K. R. Parthasarathy. *Probability Measures on Metric Spaces*. Academic Press, 1967.